

3D 横观各向同性地基非耦合解析层元

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摘 要: 基于 Fourier 变换域内直角坐标系下 3D 横观各向同性单层地基的控制方程, 借助于降阶法, 推导出单层地基非耦合的精确刚度矩阵即解析层元, 然后根据有限层法原理并引入相应的边界条件, 组装并求解总体刚度矩阵, 最后通过 Fourier 逆变换技术获得物理域内的真实解答。编制了相应的 Fortran 程序进行数值计算, 并与 ABAQUS 模拟结果进行对比以验证本文方法及结果的正确性, 计算分析结果表明: 地基的横观各向同性特性与成层特性对地基的沉降有较大的影响。

关键词: 3D 横观各向同性; 非耦合解析层元; 成层地基

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Uncoupled analytical layer-element for 3D transversely isotropic multilayered foundation

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Abstract: Based on the governing equations in Fourier transform domain for 3D transversely isotropic single-layer foundation in Cartesian coordinate system, and applying the order reduction method, an uncoupled exact stiffness matrix, i.e., the analytical layer-element of single-layer foundation, is obtained. Following the principle of the finite layer method, and introducing the corresponding boundary conditions, the total stiffness matrix is assembled and solved. In order to verify the proposed method, numerical results are obtained by the compiled procedure and compared with those by ABAQUS, which shows a good agreement. Other results demonstrate that the transversely isotropic and multilayered properties of the foundation have a great impact on its vertical displacements.

Key words: 3D transverse isotropy; uncoupled analytical layer-element; multilayered foundation

0 引 言

鉴于土体在沉积过程中存在较普遍的成层现象, 以及土体在水平向和竖向模量上存在差异(水平面内通常呈现各向同性), 将土体简化成横观各向同性体较均匀各向同性体的假设更为合理。因此, 研究成层横观各向同性地基问题具有一定的实用价值。Lekhnitskii^[1]首先求出了横观各向同性弹性体轴对称问题的通解; 胡海昌^[2]在 Lekhnitskii 的基础上求得了横观各向同性体空间问题的通解; 丁皓江^[3]对横观各向同性弹性力学进行了较为系统的研究。对于横观各向同性成层地基问题, 陈光敬等^[4-5]通过 Hankel 变换方法推导出轴对称及非轴对称成层横观各向同性地基的传递矩阵解; 艾智勇等^[6]运用二重 Fourier 变换推导

出三维横观各向同性成层地基的非耦合传递矩阵解。

与传统的传递矩阵法相比, 解析层元法^[7]中的刚度矩阵元素采用了负指数的处理方法, 避免了因正指数的存在而出现的计算溢出问题, 提高了计算稳定性。本文将解析层元法拓展应用于横观各向同性成层地基的计算分析, 并以两层土为例讨论土的横观各向同性及成层特性对地基沉降的影响。

1 非耦合解析层元的推导

弹性力学中不计体积力时用位移表示的静力平衡方程为

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$$(c_{11} \frac{\partial^2}{\partial x^2} + c_{66} \frac{\partial^2}{\partial y^2} + c_{44} \frac{\partial^2}{\partial z^2})u + (c_{12} + c_{66}) \frac{\partial^2 v}{\partial x \partial y} + (c_{13} + c_{44}) \frac{\partial^2 w}{\partial x \partial z} = 0, \quad (1a)$$

$$(c_{12} + c_{66}) \frac{\partial^2 u}{\partial x \partial y} + (c_{66} \frac{\partial^2}{\partial x^2} + c_{11} \frac{\partial^2}{\partial y^2} + c_{44} \frac{\partial^2}{\partial z^2})v + (c_{13} + c_{44}) \frac{\partial^2 w}{\partial y \partial z} = 0, \quad (1b)$$

$$(c_{13} + c_{44}) \frac{\partial}{\partial z} (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) + (c_{44} (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) + c_{33} \frac{\partial^2}{\partial z^2})w = 0. \quad (1c)$$

式中 $c_{12} = \lambda n(\mu_h + n\mu_{vh}^2)$, $c_{13} = n\lambda\mu_{vh}(1 + \mu_h)$, $c_{33} = \lambda(1 - \mu_h^2)$, $c_{44} = G_v$, $n = E_h/E_v$, $\lambda = E_v/((1 + \mu_h)(1 - \mu_h - 2n\mu_{vh}^2))$, 其中, E_h 为水平向弹性模量; E_v 为竖向弹性模量; G_v 为竖直面上的剪切模量; μ_h 为水平向应力引起的正交水平向应变的泊松比; μ_{vh} 为竖直向应力引起的水平向应变的泊松比; u, v, w 分别是 x, y, z 方向的位移。

其本构方程为

$$\tau_{xz} = c_{44} (\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}), \quad (2a)$$

$$\tau_{yz} = c_{44} (\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}), \quad (2b)$$

$$\sigma_z = c_{13} \frac{\partial u}{\partial x} + c_{13} \frac{\partial v}{\partial y} + c_{33} \frac{\partial w}{\partial z}. \quad (2c)$$

二重 Fourier 变换及其逆变换的定义为

$$\bar{f}(\xi_x, \xi_y) = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(x, y) e^{-i(\xi_x x + \xi_y y)} dx dy, \quad (3a)$$

$$g(x, y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \bar{f}(\xi_x, \xi_y) e^{i(\xi_x x + \xi_y y)} d\xi_x d\xi_y. \quad (3b)$$

式中 $g(x, y)$ 为任意函数; $\bar{f}(\xi_x, \xi_y)$ 经过双重 Fourier 变换后的函数; ξ_x, ξ_y 分别是坐标 x, y 经过双重 Fourier 变换后的量; 上面带一横表示经过双重 Fourier 变换后的量。

为了方便推导, 对位移变量作如下解耦变换^[6]:

$$\begin{cases} U = \frac{1}{\xi} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \\ V = \frac{1}{\xi} \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) \\ W = w \end{cases}, \quad (4)$$

式中, $\xi^2 = \xi_x^2 + \xi_y^2$ 。

对式 (1)、(4) 分别进行关于坐标 x, y 的双重 Fourier 变换, 并经过推导运算可得

$$\frac{\partial^2 \bar{U}}{\partial z^2} = \frac{c_{11}}{c_{44}} \xi^2 \bar{U} + \frac{c_{13} + c_{44}}{c_{44}} \xi \frac{\partial \bar{W}}{\partial z}, \quad (5a)$$

$$\frac{\partial^2 \bar{W}}{\partial z^2} = -\frac{c_{13} + c_{44}}{c_{33}} \xi \frac{\partial \bar{U}}{\partial z} + \frac{c_{44}}{c_{33}} \xi^2 \bar{W}, \quad (5b)$$

$$\frac{\partial^2 \bar{V}}{\partial z^2} = -\frac{c_{66}}{c_{44}} \xi^2 \bar{V}. \quad (5c)$$

将式 (5a)、(5b) 写成如下矩阵形式:

$$\frac{d\Theta_{4 \times 1}(\xi_x, \xi_y, z)}{dz} = A_{4 \times 4}(\xi_x, \xi_y, z) \cdot \Theta_{4 \times 1}(\xi_x, \xi_y, z). \quad (6)$$

式中, $\Theta_{4 \times 1} = [\Gamma \quad \Gamma']^T$, $\Gamma_{2 \times 1} = [\bar{U} \quad \bar{W}]^T$ 。

由 Cayley-Hamilton 定理可知, 式 (6) 可以表示为

$$\Theta(\xi_x, \xi_y, z) = B(\xi_x, \xi_y, z) \cdot \Theta(\xi_x, \xi_y, 0). \quad (7)$$

式中, $B(\xi_x, \xi_y, z) = \exp[zA(\xi_x, \xi_y, z)] = a_0 I + \sum_{i=1}^3 a_i A^i$,

其中, $a_i (i=0, \dots, 3)$ 是与 $A(\xi_x, \xi_y, z)$ 有关的量, 而 $A(\xi_x, \xi_y, z)$ 的特征值满足下列方程:

$$\lambda_1^4 + (2d_1 - d_2 d_4) \xi^2 \lambda_1^2 + (d_1^2 + d_2 d_3) \xi^4 = 0. \quad (8)$$

为解特征方程 (8), 令 $\alpha^2 = ((d_2 d_4 - 2d_1) + 2\sqrt{d_1^2 + d_2 d_3})/4$, $\beta^2 = ((d_2 d_4 - 2d_1) - 2\sqrt{d_1^2 + d_2 d_3})/4$, 则 $\lambda_1^2 = (\alpha^2 + \beta^2) \xi^2 \pm \sqrt{4\alpha^2 \beta^2} \xi^2$, 其中, $d_1 = c_{13}/c_{33}$, $d_2 = (c_{11} c_{33} - c_{13}^2)/c_{33}$, $d_3 = 1/c_{33}$, $d_4 = 1/c_{44}$, $d_5 = (c_{11} - c_{12})/2$ 。

由于 β^2 正负的不确定性, 决定了对它的符号进行讨论, 于是有: ①当 $\beta^2 = 0$ 时, $\lambda_1 = \pm \alpha \beta$; ②当 $\beta^2 > 0$ 时, 令 $\rho = \alpha + \beta$, $\phi = \alpha - \beta$, 则 $\lambda_1 = \pm \rho \xi$ 或 $\lambda_1 = \pm \phi \xi$; ③当 $\beta^2 < 0$ 时, 令 $\omega^2 = -\beta^2$, 则 $\lambda_1 = \pm (\alpha \pm \omega i) \xi$ 。

由以上特征值及矩阵 $A(\xi_x, \xi_y, z)$, 可得矩阵 $B(\xi_x,$

$$\xi_y, z), \text{ 即 } B(\xi_x, \xi_y, z) = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}, \text{ 则 } \Theta(\xi_x, \xi_y, z) =$$

$$\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} \cdot C, \text{ 其中 } C = \Theta(\xi_x, \xi_y, 0).$$

于是可得

$$\begin{bmatrix} \Gamma(\xi_x, \xi_y, 0) \\ \Gamma(\xi_x, \xi_y, z) \end{bmatrix} = \begin{bmatrix} I & 0 \\ B_1 & B_2 \end{bmatrix} \cdot C. \quad (9)$$

为了方便推导, 对应力变量作如下解耦变换^[6]:

$$\begin{cases} X = \frac{1}{\xi} \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} \right) \\ Y = \frac{1}{\xi} \left(\frac{\partial \tau_{xz}}{\partial y} - \frac{\partial \tau_{yz}}{\partial x} \right) \\ Z = \sigma_z \end{cases}, \quad (10)$$

结合式 (2)、(3) 和 (10), 于是可得

$$\begin{bmatrix} -\Omega(\xi_x, \xi_y, 0) \\ \Omega(\xi_x, \xi_y, z) \end{bmatrix} = \begin{bmatrix} -DB(\xi_x, \xi_y, 0) \\ DB(\xi_x, \xi_y, z) \end{bmatrix} \cdot C, \quad (11)$$

式中,

$$\Omega(\xi_x, \xi_y, z) = [\bar{X} \quad \bar{Y}]^T, D = \begin{bmatrix} 0 & -\frac{\xi}{d_4} & \frac{1}{d_4} & 0 \\ \frac{d_1 \xi}{d_3} & 0 & 0 & \frac{1}{d_3} \end{bmatrix}.$$

由式 (9)、(11) 可得

$$\begin{bmatrix} -\Omega(\xi_x, \xi_y, 0) \\ \Omega(\xi_x, \xi_y, z) \end{bmatrix} = K \cdot \begin{bmatrix} \Theta(\xi_x, \xi_y, 0) \\ \Theta(\xi_x, \xi_y, z) \end{bmatrix} \quad (12)$$

式中, $K = \begin{bmatrix} -DB(\xi_x, \xi_y, 0) \\ DB(\xi_x, \xi_y, z) \end{bmatrix} \cdot \begin{bmatrix} I & 0 \\ B_1 & B_2 \end{bmatrix}^{-1}$, 它反映了 $z=0$ 和任意深度 z 处位移和应力之间的关系, 其具体元素可见附录。

同理, 可以求得

$$\begin{bmatrix} \bar{Y}(\xi_x, \xi_y, 0) \\ \bar{Y}(\xi_x, \xi_y, z) \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} \bar{V}(\xi_x, \xi_y, 0) \\ \bar{V}(\xi_x, \xi_y, z) \end{bmatrix} \quad (13)$$

式中, $T_{11}, T_{12}, T_{21}, T_{22}$ 具体见附录。

2 成层地基非耦合解析层元解

基于有限层法原理, 可以得到 n 层地基的总刚度矩阵方程:

$$\begin{bmatrix} K^1 \\ K^2 \\ \vdots \\ K^{n-1} \\ K^n \end{bmatrix} \begin{bmatrix} \bar{U}(\xi_x, \xi_y, 0) \\ \bar{W}(\xi_x, \xi_y, 0) \\ \vdots \\ \bar{U}(\xi_x, \xi_y, z_n) \\ \bar{W}(\xi_x, \xi_y, z_n) \end{bmatrix} = \begin{bmatrix} -\bar{X}(\xi_x, \xi_y, 0) \\ -\bar{Z}(\xi_x, \xi_y, 0) \\ \vdots \\ -\bar{X}(\xi_x, \xi_y, z_n) \\ -\bar{Z}(\xi_x, \xi_y, z_n) \end{bmatrix} \quad (14)$$

式中 K^i 为第 i 层土体的层元刚度矩阵。

根据边界条件解式 (14), 可以得到在 Fourier 变换域里的位移解, 利用编制的 Fortran 程序对位移进行 Fourier 数值逆变换, 便可得到其在物理域内的精确解。

3 数值计算与分析

本文根据 10 点复合二维高斯积分的方法, 在被积函数 2 个零点之间进行分段积分, 实现 Fourier 数值逆变换^[6]。为了验证本文方法及结果的正确性, 将 Fortran 程序计算获得的竖向荷载作用下任意深度的沉降量, 与 ABAQUS 模拟的结果进行比较, 具体见图 1。由图 1 可知, 本文的计算结果与 ABAQUS 模拟结果比较吻合, 从而验证了本文理论和方法的正确性。

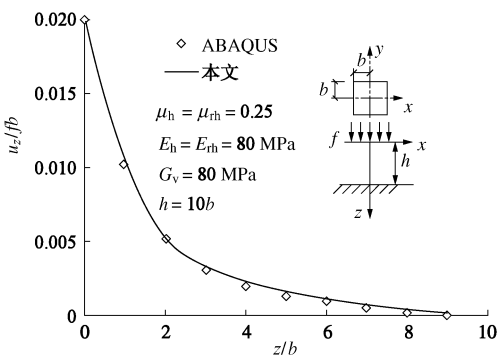


图 1 竖向位移对比结果
Fig. 1 Comparison of vertical displacements

下面以两层土为例讨论土的横观各向同性及成层特性对地基沉降的影响。土层参数见表 1, 表 1 中的工况 1 为上层软土下层硬土, 工况 2 为上层硬土下层软土, 而工况 3 为工况 1 和工况 2 各参数的加权平均值, 是一各向同性的均匀土层。计算结果见图 2, 由图 2 可知: 在相同的均布荷载作用下, 不同分层地基中的竖向位移有较显著的差别; 另外, 无论是对于工况 1 还是工况 2, 采用传统均匀各向同性地基模型简化计算的结果均偏小, 因此实际工程中应采用更加符合实际情况的横观各向同性成层地基模型。

表 1 土层参数

工况	土层一			土层二			$\mu_{vh} = \mu_h$
	G_v /MPa	E_v /MPa	E_h /MPa	G_v /MPa	E_v /MPa	E_h /MPa	
1	10	20	20	78	200	200	0.25
2	78	200	200	10	20	20	0.25
3	44	110	110	44	110	110	0.25

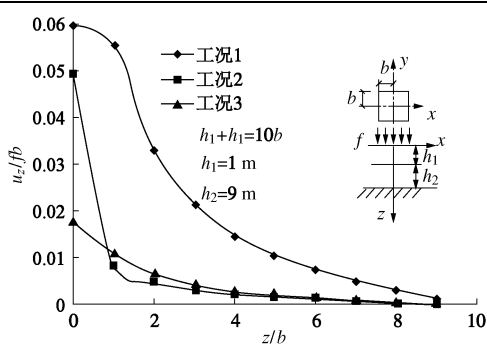


图 2 两层地基中的竖向位移
Fig. 2 Vertical displacements of two-layered soils

4 结 语

从弹性力学的基本方程出发, 采用解耦变换并利用双重 Fourier 变换以及 Cayley-Hamilton 定理, 借助于降阶法, 推导出直角坐标系下横观各向同性单层地基的非耦合解析层元, 然后根据有限层法原理并引入相应的边界条件, 组装得到总体刚度矩阵, 最后通过 Fourier 逆变换技术获得物理域内的真实解答。将所得结果和 ABAQUS 模拟的结果进行了对比, 验证了本文的正确性; 以两层土为例讨论土的横观各向同性及成层特性对地基沉降的影响, 结果表明: 传统均匀各向同性地基模型简化计算的结果与实际差别较大, 地基的成层特性对其竖向位移有较显著的影响; 因此, 实际工程中应采用更加符合实际情况的横观各向同性成层地基模型。

必须指出, 本文分析均假设土体为弹性体, 但实际土体经常表现出弹塑性的性质, 甚至为弹黏塑性体; 为此, 在后续的研究中, 笔者将就相关问题作进一步的探讨。

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附录: 3 种情况下的解析层元元素 (该矩阵为对称矩阵)

(1) $\beta^2 = 0$ 时

$$k_{11} = 4\xi(-2d_1 + d_2d_4)(\alpha A_1(8d_1 - 3d_2d_4) + 2z\xi d_2d_4 e^{-2z\alpha\xi} \cdot (-2d_1 + d_2d_4))/[d_4(-A_3 + 2A_4)]; \quad k_{12} = -(2\xi(-2A_4 + A_2^2(8d_1 - 3d_2d_4)(4d_1 - d_2d_4)))/[d_4(-A_3 + 2A_4)]; \quad k_{13} =$$

$$8e^{-z\alpha\xi}\alpha\xi(-2d_1 + d_2d_4)(8A_2d_1 + (-3 + z\alpha\xi + e^{-2z\alpha\xi}(3 + z\alpha\xi))d_2d_4)/[d_4(A_3 - 2A_4)]; \quad k_{14} = -8z\alpha\xi^2 A_2d_2e^{-z\alpha\xi}(-2d_1 + d_2d_4)/(-A_3 + 2A_4); \quad k_{22} = 8\xi(\alpha A_1(8d_1 - 3d_2d_4) - 2z\xi d_2d_4 e^{-2z\alpha\xi} \cdot (-2d_1 + d_2d_4))/(d_4(-A_3 + 2A_4)); \quad k_{24} = 16e^{-z\alpha\xi}\alpha\xi(-8A_2d_1 + (3 + z\alpha\xi + e^{-2z\alpha\xi}(-3 + z\alpha\xi))d_2d_4)/(d_4(-3A_3 + 2A_4)); \quad k_{23} = -k_{14}; \quad k_{33} = k_{11}; \quad k_{34} = -k_{12}; \quad k_{44} = k_{22}.$$

其中: $A_1 = 1 - e^{-4z\alpha\xi}$; $A_2 = 1 - e^{-2z\alpha\xi}$; $A_3 = A_2^2(8d_1 - 3d_2d_4)^2$; $A_4 = z^2\xi^2 d_2^2 d_4^2 e^{-2z\alpha\xi}(-2d_1 + d_2d_4)$.

(2) $\beta^2 > 0$ 时

$$k_{11} = \xi(-\rho\phi A_1(\phi A_3 A_7 - \rho A_2 A_8)(d_3 + d_1d_4) + (A_1 A_8 + A_4 A_9) \cdot ((\rho A_3(\rho^2 + 2d_1) - \phi A_2(\phi^2 + 2d_1))d_3 + (-\phi A_2 + \rho A_3)d_1^2 d_4))/ (A_5 A_6 + \rho\phi A_1^2(d_3 + d_1d_4)^2); \quad k_{12} = \xi(A_5(\phi A_2 A_7 - \rho A_3 A_8) + \rho\phi A_1(-A_9 A_4 + A_1 A_7)(d_3 + d_1d_4))/(A_5 A_6 + \rho\phi A_1^2(d_3 + d_1d_4)^2); \quad k_{13} = -2e^{-z\xi(\rho+\phi)}\xi\rho\phi A_3 A_9/(A_5 A_6 + \rho(d_3 + d_1d_4)^2\phi A_1^2); \quad k_{14} = 2e^{-z\xi(\rho+\phi)}\xi\rho\phi A_1 A_9(d_3 + d_1d_4)/(A_5 A_6 + \rho(d_3 + d_1d_4)^2\phi A_1^2); \quad k_{22} = \xi(A_6(-A_1 A_7 + A_4 A_9) + A_1(\phi A_2 A_7 - \rho A_3 A_8)(d_3 + d_1d_4))/(A_5 A_6 + \rho\phi A_1^2(d_3 + d_1d_4)^2); \quad k_{24} = -2e^{-z\xi(\rho+\phi)} \cdot \xi A_6 A_9/(A_5 A_6 + \rho\phi A_1^2(d_3 + d_1d_4)^2); \quad k_{23} = -k_{14}; \quad k_{33} = k_{11}; \quad k_{34} = -k_{12}; \quad k_{44} = k_{22}.$$

其中: $A_1 = e^{-z\xi(2\rho+\phi)} + e^{-z\xi\phi} - e^{-z\xi(\rho+2\phi)} - e^{-z\xi\rho}$; $A_2 = -e^{-z\xi(2\rho+\phi)} + e^{-z\xi\phi}$; $A_3 = -e^{-z\xi(\rho+2\phi)} + e^{-z\xi\rho}$; $A_4 = e^{-z\xi(2\rho+\phi)} + e^{-z\xi\phi}$; $A_5 = (\rho A_2 - \phi A_3)d_3 + \rho\phi(-\phi A_2 + \rho A_3)d_4$; $A_6 = (-\phi A_2 + \rho A_3)d_3 + \rho\phi(\rho A_2 - \phi A_3)d_4$; $A_7 = \rho^2 + d_1$; $A_8 = \phi^2 + d_1$; $A_9 = \rho^2 - \phi^2$.

(3) $\beta^2 < 0$ 时

$$k_{11} = \xi A_3(-2A_1 A_5 S_1(-\alpha A_2(A_3 + d_1)S_1 + (\alpha A_2(d_3 - A_3d_4) \cdot S_1 + \omega A_1 A_6 S_2)/(2A_4) - \omega A_1(A_3 - d_1)S_2) + (A_1 d_2 d_4 S_1 + 4\alpha\omega A_2 S_2)), \quad k_{12} = \xi((A_1^2 A_3 A_5(\alpha^2 - \omega^2 + d_1) + \alpha^2 A_2^2(A_3 + d_1)(d_3 - A_3d_4))S_1^2 + \omega^2 A_1^2(A_3 - d_1)A_6 S_2^2)/A_4; \quad k_{13} = -4e^{-z\alpha\xi}\alpha\xi\omega A_3(\alpha A_2(d_3 - A_3d_4)S_1 + \omega A_1 A_6 S_2)/A_4; \quad k_{14} = 4e^{-z\alpha\xi}\alpha\xi\omega A_1 A_3 A_5 S_1/A_4; \quad k_{22} = \xi(-A_1 A_5 S_1(-\alpha A_2(d_1 + A_3)S_1 + \omega A_1(d_1 - A_3)S_2) + (-A_1(\alpha^2 - \omega^2 + d_1)S_1 + 2\alpha\xi A_2 S_2) - (\alpha A_2(-d_3 + A_3d_4)S_1 + \omega A_1 A_6 S_2))/A_4; \quad k_{24} = -4e^{-z\alpha\xi}\alpha\xi\omega(\alpha A_2(-d_3 + A_3d_4)S_1 + \omega A_1 A_6 S_2)/A_4; \quad k_{23} = -k_{14}, \quad k_{33} = k_{11}, \quad k_{34} = -k_{12}, \quad k_{44} = k_{22}.$$

其中: $A_1 = 1 - e^{-2z\alpha\xi}$; $A_2 = 1 + e^{-2z\alpha\xi}$; $A_4 = -\alpha^2 A_2^2(d_3 - A_3d_4)^2 S_1^2 + A_1^2(A_3 A_7 S_1^2 + \omega^2 A_8 S_2^2)$; $A_3 = \alpha^2 + \omega^2$; $A_5 = d_3 + d_1d_4$; $A_6 = d_3 + A_3d_4$; $S_1 = \sin[z\xi\omega]$; $S_2 = \cos[z\xi\omega]$.

$$T_{11} = -\frac{(1 + e^{-2\gamma\xi z})\gamma z}{(1 - e^{-2\gamma\xi z})d_4}, \quad T_{12} = \frac{2\gamma\xi e^{-\gamma\xi z}}{(1 - e^{-2\gamma\xi z})d_4}, \quad T_{21} = \frac{(1 - e^{-2\gamma\xi z})d_5}{2\gamma^2 d_4 e^{-\gamma\xi z}}, \quad T_{22} = \frac{(1 + e^{-2\gamma\xi z})\gamma z}{(1 - e^{-2\gamma\xi z})d_4}.$$

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