

解析层元法求解层状横观各向同性地基轴对称问题

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摘要: 提出了一个求解横观各向同性层状地基轴对称问题的解析层元方法。从弹性力学基本方程出发并利用 Hankel 变换, 得到横观各向同性单层地基的传递矩阵解, 进而推导出单层地基的解析层元刚度矩阵。利用有限元方法组装总体刚度矩阵, 通过求解总体刚度矩阵, 并采用 Hankel 逆变换的数值积分方法, 可求出层状横观各向同性弹性体轴对称问题在物理域内的精确解。刚度矩阵元素中不存在正指数并具有对称的特点, 不仅使计算过程简化, 还提高了计算精度。最后文中给出了算例来证明推导结果的准确性。

关键词: 横观各向同性; 轴对称问题; 解析层元; 成层土体

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Analytical layer-element method for axisymmetric problem of transversely isotropic multi-layered soils

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Abstract: An analytical layer-element method is proposed to solve the axisymmetric problem of transversely isotropic multi-layered soils. From the basic equations of elasticity and by using the Hankel transform, the transfer matrix of a single soil layer is obtained. Based on the transfer matrix, a precise stiffness matrix of a single soil layer is deduced. By employing the finite element method, we can further get the total stiffness matrix of multi-layered soils. A numerical inversion technique of the Hankel transform is adopted to obtain the exact solutions of axisymmetric problems of transversely isotropic multi-layered soils in the physical domain. Because of no existence of positive functions and the symmetry of the matrix, we can not only simplify the calculation but also improve its accuracy. Finally, a numerical example is given to prove the accuracy of the present results.

Key words: transverse isotropy; axisymmetric problem; analytical layer-element; multi-layered soil

0 引言

天然土体由于长期沉积形成, 往往成层分布并呈现出横观各向同性性质。因此, 横观各向同性与均匀各向同性相比更接近于实际土体的性质。实际上, 如重超固结伦敦黏土、轻超固结高岭黏土、正常固结伊利黏土、正常固结黏土夹粉土及 Colorado 黏土等地基, 均体现了横观各向同性地基的性质, 这些地基如果按照均匀各向同性地基考虑, 可能引起建筑物基础的不均匀沉降, 危及建筑物安全。国内外有许多学者对此作了相应的研究, Lekhnitskii^[1]首先导出了一个横观各向同性弹性体轴对称问题的通解, 顿林志等^[2]、胡雨晨等^[3]从胡海昌解^[4]出发, 分别推导了层状横观各向

同性地基在轴对称荷载作用下的解析解, 陈光敬等^[5]通过 Hankel 变换方法得到了柱坐标下成层横观各向同性地基的传递矩阵解, 唐和生等^[6]利用常微分方程求解原理, 直接得出了在轴对称横观各向同性半无限弹性空间的位移, 但没有给出具体的刚度矩阵元素。

类似于文献[7], 本文从单层地基的传递矩阵推导出横观各向同性土体的刚度矩阵, 然后根据有限元方法构造总体刚度矩阵, 将 Hankel 逆变换的数值积分方法应用于 Fortran 编程, 求解了在静力荷载作用下横观

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各向同性土体的位移。由于刚度矩阵的对称性以及其元素仅存在负指数这一优点, 不仅避免了因正指数存在导致的计算溢出问题, 提高计算精度, 也在很大程度上简化了整个计算过程。

1 层元刚度矩阵的推导

空间轴对称问题的静力平衡方程为

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0, \quad (1)$$

$$\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{rz}}{\partial r} + \frac{\tau_{rz}}{r} = 0. \quad (2)$$

用位移表示的应力应变关系为

$$\sigma_r = a \frac{\partial u}{\partial r} + b \frac{u}{r} + c \frac{\partial w}{\partial z}, \quad (3)$$

$$\sigma_\theta = b \frac{\partial u}{\partial r} + a \frac{u}{r} + c \frac{\partial w}{\partial z}, \quad (4)$$

$$\sigma_z = c \frac{\partial u}{\partial r} + c \frac{u}{r} + d \frac{\partial w}{\partial z}, \quad (5)$$

$$\tau_{rz} = \frac{f}{2} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right). \quad (6)$$

式中, $a = \lambda n(1 - n\mu_{vh}^2)$, $b = \lambda n(\mu_h + n\mu_{vh}^2)$, $c = \lambda n\mu_{vh}(1 + \mu_h)$, $d = \lambda(1 - \mu_h^2)$, $f/2 = G_v$, $\lambda = E_v / ((1 + \mu_h)(1 - \mu_h - 2n\mu_{vh}^2))$, $n = E_h/E_v$ 。其中: E_h 为水平向弹性模量; E_v 为竖向弹性模量; G_v 为竖直面上的剪切模量; μ_h 为水平向应力引起的正交水平向应变的泊松比; μ_{vh} 为竖直面应力引起的水平向应变的泊松比。

由文献[5]可知, 根据式(1)~(6)并进行 Hankel 变换^[8]可推出

$$\frac{d}{dz} \begin{Bmatrix} \bar{u}(\xi, z) \\ \bar{w}(\xi, z) \\ \bar{\sigma}_z(\xi, z) \\ \bar{\tau}_{rz}(\xi, z) \end{Bmatrix} = \begin{bmatrix} 0 & \xi & 0 & \frac{2}{f} \\ -\frac{c}{d}\xi & 0 & \frac{1}{d} & 0 \\ 0 & 0 & 0 & -\xi \\ \frac{ad-c^2}{d}\xi^2 & 0 & \frac{c}{d}\xi & 0 \end{bmatrix} \begin{Bmatrix} \bar{u}(\xi, z) \\ \bar{w}(\xi, z) \\ \bar{\sigma}_z(\xi, z) \\ \bar{\tau}_{rz}(\xi, z) \end{Bmatrix}. \quad (7)$$

式中, ξ 为相应的 Hankel 变换参数, $\bar{u}(\xi, z)$ 、 $\bar{\tau}_{rz}(\xi, z)$ 为 $u(r, z)$ 、 $\tau_{rz}(r, z)$ 进行 1 阶 Hankel 变换后的量, $\bar{w}(\xi, z)$ 、 $\bar{\sigma}_z(\xi, z)$ 为 $w(r, z)$ 、 $\sigma_z(r, z)$ 进行 0 阶 Hankel 变换后的量。

令向量 $\bar{X}(\xi, z) = [\bar{u}, \bar{w}, \bar{\sigma}_z, \bar{\tau}_{rz}]^T$, 则式(7)可以转换为

$$\frac{d}{dz} [\bar{X}(\xi, z)] = A(\xi) \bar{X}(\xi, z). \quad (8)$$

求解式(8)可以得到

$$\bar{X}(\xi, z) = \exp[zA(\xi)] \bar{X}(\xi, 0). \quad (9)$$

式(9)中的 $\exp[zA(\xi)]$ 可由 Cayley-Hamilton 原理求解, 并且 $A(\xi)$ 的特征方程可以表示为

$$\lambda^4 + \frac{2(cf - ad + c^2)}{df} \xi^2 \lambda^2 + \frac{a}{d} \xi^4 = 0. \quad (10)$$

$$\text{令 } \alpha^2 = \frac{ad - c^2 - cf + f\sqrt{ad}}{2df}, \quad \beta^2 = \frac{ad - c^2 - cf - f\sqrt{ad}}{2df},$$

那么式(10)的解可以表示成

$$\lambda^2 = \left(\frac{ad - c^2 - cf}{df} \pm \sqrt{\left(\frac{ad - c^2 - cf}{df} \right)^2 - \frac{a}{d}} \right) \xi^2 \\ = (\alpha^2 + \beta^2) \xi^2 \pm \sqrt{4\alpha^2 \beta^2} \xi^2. \quad (11)$$

式(9)矩阵形式为

$$\begin{Bmatrix} \bar{u}(\xi, z) \\ \bar{w}(\xi, z) \\ \bar{\sigma}_z(\xi, z) \\ \bar{\tau}_{rz}(\xi, z) \end{Bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} & T_{14} \\ T_{21} & T_{22} & T_{23} & T_{24} \\ T_{31} & T_{32} & T_{33} & T_{34} \\ T_{41} & T_{42} & T_{43} & T_{44} \end{bmatrix} \begin{Bmatrix} \bar{u}(\xi, 0) \\ \bar{w}(\xi, 0) \\ \bar{\sigma}_z(\xi, 0) \\ \bar{\tau}_{rz}(\xi, 0) \end{Bmatrix}, \quad (12)$$

式中, T_{ij} 为传递矩阵元素。

式(12)可转化为

$$\begin{Bmatrix} -\bar{\tau}_{rz}(\xi, 0) \\ -\bar{\sigma}_z(\xi, 0) \\ \bar{\tau}_{rz}(\xi, z) \\ \bar{\sigma}_z(\xi, z) \end{Bmatrix} = \begin{bmatrix} K1 & K2 \\ K3 & K4 \end{bmatrix} \begin{Bmatrix} \bar{u}(\xi, 0) \\ \bar{w}(\xi, 0) \\ \bar{u}(\xi, z) \\ \bar{w}(\xi, z) \end{Bmatrix}, \quad (13)$$

$$\text{式中, } K1 = \begin{bmatrix} T_{14} & T_{13} \\ T_{24} & T_{23} \end{bmatrix}^{-1} \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}, \quad K2 = - \begin{bmatrix} T_{14} & T_{13} \\ T_{24} & T_{23} \end{bmatrix}^{-1},$$

$$K4 = \begin{bmatrix} T_{44} & T_{43} \\ T_{34} & T_{33} \end{bmatrix} \begin{bmatrix} T_{14} & T_{13} \\ T_{24} & T_{23} \end{bmatrix}^{-1},$$

$$K3 = \begin{bmatrix} T_{41} & T_{42} \\ T_{31} & T_{32} \end{bmatrix} - \begin{bmatrix} T_{44} & T_{43} \\ T_{34} & T_{33} \end{bmatrix} \begin{bmatrix} T_{14} & T_{13} \\ T_{24} & T_{23} \end{bmatrix}^{-1} \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}.$$

$$\begin{Bmatrix} -\bar{\tau}_{rz}(\xi, 0) \\ -\bar{\sigma}_z(\xi, 0) \\ \bar{\tau}_{rz}(\xi, z) \\ \bar{\sigma}_z(\xi, z) \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ & k_{22} - k_{14} & k_{24} & \\ \text{对 称} & & k_{11} - k_{12} & \\ & & & k_{22} \end{bmatrix} \begin{Bmatrix} \bar{u}(\xi, 0) \\ \bar{w}(\xi, 0) \\ \bar{u}(\xi, z) \\ \bar{w}(\xi, z) \end{Bmatrix}.$$

根据式(11)中的 β^2 的取值情况, 可以分 3 种情况, 具体元素见附录。

2 层状横观各向同性土体的求解

如图 1 所示, 层状土体在其表面作用一均布圆形荷载 p , 荷载半径为 a , h_i , h_{i-1} 分别为第 i 层土层底面和顶面距离地表的深度, $\Delta h_i = h_i - h_{i-1}$ 为第 i 层土层的厚度。

根据上述的层元刚度矩阵, 利用有限元的方法, 建立层状土体的总刚度矩阵如下:

$$\begin{pmatrix} K^1 & & & \\ & K^2 & & \\ & & \ddots & \\ & & & K^{m-1} \\ & & & & K^m \end{pmatrix} \begin{bmatrix} u(\xi, 0) \\ w(\xi, 0) \\ \vdots \\ u(\xi, z_m) \\ w(\xi, z_m) \end{bmatrix} = \begin{bmatrix} \tau_{rz}(\xi, 0) \\ \sigma_z(\xi, 0) \\ \vdots \\ \tau_{rz}(\xi, z_m) \\ \sigma_z(\xi, z_m) \end{bmatrix}, \quad (14)$$

式中, K^i 为第 i 层土体的层元刚度矩阵。

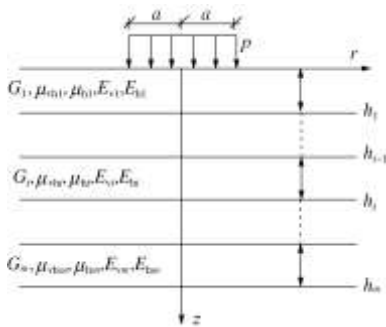


图 1 轴对称荷载作用下层状横观各向同性地基
Fig. 1 Transversely isotropic multi-layered soils under axisymmetric loading

将层间接触面的位移协调条件及底面固定的边界条件, 代入式 (14) 并求解, 可以得到层状横观各向同性地基轴对称问题在 Hankel 变化域内的解答, 利用 Fortran 程序对其进行 Hankel 逆变换, 可求出物理域内的解。

3 数值计算与分析

3.1 理论验证

为了验证本文解答的正确性, 按本文方法编制计算机程序, 计算均布圆形荷载作用下的层状地基中任意深度的位移, 并与文献[9]及 Abaqus 的计算结果进行比较。图 2 为本文退化为各向同性地基的结果, 其计算结果与文献[9]的结果吻合较好; 图 3 为横观各向同性地基计算结果, 与 Abaqus 计算结果吻合也较好; 从而验证了本文的正确性。

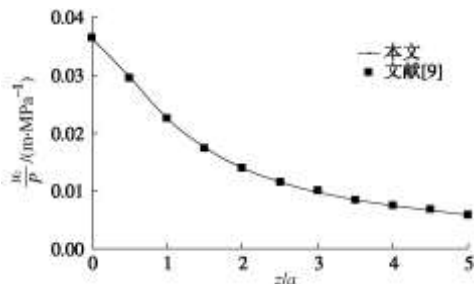


图 2 退化后的地基竖向位移比较

Fig. 2 Comparison of vertical displacements of simplified soils

3.2 地基的横观各向同性特性对竖向位移的影响

此算例研究地基的水平向弹性模量 E_h 与竖向弹性模量 E_v 的比值 n 对竖向位移的影响。图 4 表明, 横观各向同性地基表面作用均布圆形荷载引起的竖向位移随 n 增大而减小, 且计算点越接近于表面, 产生的竖向位移的差距越显著。可见, 传统的均匀各向同性地基模型 ($n = 1.0$) 并不能全面有效地反映土体特性; 而将横观各向同性地基模型应用于地基基础的设计计算具有一定的意义。

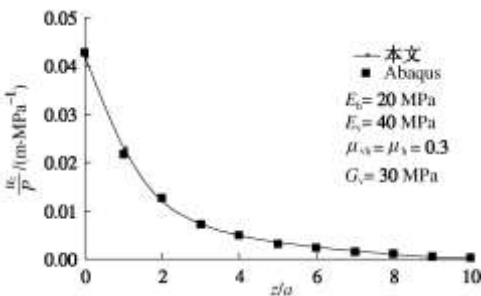


图 3 横观各向同性地基竖向位移比较
Fig. 3 Comparison of vertical displacements of transversely isotropic soils

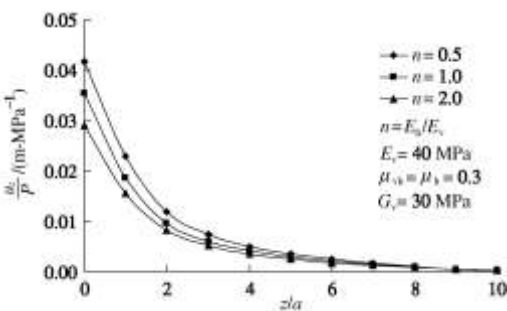


图 4 不同 n 值对竖向位移的影响
Fig. 4 Variation of vertical displacements due to different values of parameter n

3.3 土的分层特性对竖向位移的影响

为了进一步研究土体分层特性对竖向位移的影响, 本文分别对不同土层参数情况下地基沿深度方向的竖向位移进行计算, 计算时的各种工况如表 1 所示。计算时土层总厚度为 $h = 10a$, 且各土层的泊松比均为 $\mu_{vh} = \mu_h = 0.3$ 。

表 1 计算工况一览表

Table 1 List of different cases							
工 况	G_v /MPa	土层 1		土层 2		土层 3	
		E_v /MPa	E_h /MPa	E_v /MPa	E_h /MPa	E_v /MPa	E_h /MPa
1	30	40	60	20	10	40	40
2	30	40	60	40	20	40	40
3	30	40	60	80	40	40	40

计算结果如图 5 所示, 图 5 表明, 在上层与底层的土体参数不变的情况下, 中间层竖向弹性模量愈大 ($n = 0.5$ 不变), 上层土的竖向位移愈小; 3 种工况产生的竖向位移在土层交界处差距最大, 且由于固定边界的假设, 竖向位移值最终都趋向于零。此算例说明, 为尽可能反映地基受力时的位移分布, 考虑实际地基的成层特点是十分必要的。

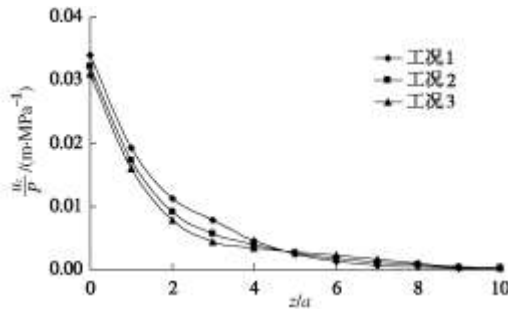


图 5 不同工况下地基竖向位移的变化

Fig. 5 Vertical displacements of multi-layered soils under different cases

4 结 语

从弹性力学基本方程出发并利用 Hankel 变换, 得到了横观各向同性单层地基的传递矩阵解, 进而推导出单层地基的解析层元刚度矩阵。利用有限元方法组装总体刚度矩阵, 通过求解总体刚度矩阵, 并采用 Hankel 逆变换的数值积分方法, 可求出层状横观各向同性弹性体轴对称问题在物理域内的解。解析层元刚度矩阵元素中不存在正指数并具有对称的特点, 不仅使计算过程简化, 还提高了计算精度。通过与已发表文献的结果进行对比, 验证了本文方法和程序的正确性。层状横观各向同性地基竖向位移的计算结果表明: 横观各向同性地基的水平向弹性模量 E_h 与竖向弹性模量 E_v 的比值 n 的变化以及地基的分层特性对于地基的竖向位移具有较为显著的影响。

由于横观各向同性地基模型比均匀各向同性地基模型更能反映实际土体特性, 所以研究成果具有一定的理论和工程应用价值。

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附录:

(1) 当 $\beta^2 = 0$ 时: $k_{11} = dA_5\xi((A_5 - 2A_7)A_1\alpha A_7 + A_6z\xi)/B_1$, $k_{12} = \xi(A_3^2A_7(c^2A_8 + A_7^3A_9) - z^2\xi^2(A_7A_5^2A_6^2\alpha^2A_4^2 - A_{10})) / B_2$, $k_{13} = 4dA_5^2e^{-z\alpha\xi}\alpha^3\xi(A_7^2(3 + 2\alpha\xi)e^{-2z\alpha\xi} - 3 + 2\alpha\xi + A_{16}) / B_2$, $k_{14} = 4d^2A_5^2A_6e^{-z\alpha\xi}A_3z\alpha^5\xi^2 / B_2$, $k_{22} = 2A_7A_5\xi(-d(A_5 - 2A_7)A_5A_1\alpha + A_5A_6z\xi A_{17}) / B_2$, $k_{24} = 4d^2A_5^2\alpha^3\xi e^{-z\alpha\xi}((A_5 - 2A_7)A_1 - A_6A_2z\alpha\xi) / B_2$;

$$B_1 = A_{18} - e^{-2z\alpha\xi}(c^2A_{14} - 9dA_7^3 + 2A_7^4z^2\xi^2) + A_{15}, \quad B_2 = A_7A_5A_6^2A_4^2z^2\alpha^2\xi^2 + A_3^2(9A_7^4 + A_6A_7^2dz^2\xi^2 - c^3A_7 - A_{11}).$$

其中, $A_1 = -e^{-4z\alpha\xi} + 1, A_2 = e^{-4z\alpha\xi} + 1, A_3 = -e^{-2z\alpha\xi} + 1, A_4 = e^{-2z\alpha\xi} + 1, A_5 = c - A_7, A_6 = c + A_7, A_7 = \sqrt{ad}, A_8 = (6A_7^2 + c^2 - 6A_7A_5), A_9 = (az^2\xi^2 - 7c + 3A_5), A_{10} = ac^2(2A_3^2A_7^2 + c^2), A_{11} = c(8A_7^3 - 7A_7^2A_5 + acz^2\xi^2A_6), A_{12} = (15d - 2A_7z^2\xi^2), A_{13} = c^3(d + 2A_7z^2\xi^2), A_{14} = (-7dA_7 + 2A_7^2z^2\xi^2), A_{15} = 0.5d(cA_5 + 4A_7^2 - 5A_7^2A_5)A_5A_2, A_{16} = cA_7(2\alpha\xi A_4 + A_3), A_{17} = (dA_4^2\alpha^2 - A_7A_3^2), A_{18} = (e^{-2z\alpha\xi}cA_7^2A_{12} + A_{13}).$

(2) 当 $\beta^2 > 0$ 时, 令 $\rho = \alpha + \beta, \phi = \alpha - \beta$, 则

$$k_{11} = A_{29}(e^{-z\xi(\rho+\phi)}A_{18} + e^{-2z\xi\phi}A_{19} + e^{-2z\xi\rho}A_{20} + e^{-z\xi(\phi+\rho)}A_{21})/B_1,$$

$$k_{12} = f\xi(d\rho^2\phi^2A_8fA_{15}(e^{-z\xi\phi}A_{10}A_{16} - e^{-z\xi\rho}A_{12}A_{17}) + A_{24}A_{25})/B_2,$$

$$k_{13} = 2(\rho^2 - \phi^2)A_{29}e^{-z\xi(\phi+\rho)}(-e^{-z\xi\rho}A_{11}\rho A_2 + e^{-z\xi\phi}A_9\phi A_1)/B_2,$$

$$k_{14} = 2(\rho^2 - \phi^2)\rho^2\phi^2d^2e^{-z\xi(\rho+\phi)}(e^{-z\xi\phi}A_{10} - e^{-z\xi\rho}A_{12})f^2A_{15}\xi/B_2,$$

$$k_{22} = df\xi\rho\phi(A_8A_{15}(e^{-z\xi\rho}A_{11}\rho A_6 - e^{-z\xi\phi}A_9\phi A_7) - A_{26}A_{27})/B_2,$$

$$k_{24} = 2\rho\phi(\rho^2 - \phi^2)d^2f\xi e^{-z\xi(\rho+\phi)}(e^{-z\xi\phi}A_9\phi A_5 - e^{-z\xi\rho}A_{11}\rho A_4)/B_2;$$

$$B_1 = A_3(de^{-2z\xi\phi}A_{10}^2 - 2de^{-z\xi(\rho+\phi)}A_{10}A_{12} + de^{-2z\xi\rho}A_{12}^2) + A_{23}A_{22},$$

$$B_2 = d\rho^2\phi^2A_8fA_{15}^2 + A_{28}(-e^{-z\xi\rho}A_{11}A_6\rho + e^{-z\xi\phi}A_9\phi A_7).$$

其中, $A_1 = 2c^2 + 2cf + df\phi^2, A_2 = 2c^2 + 2cf + df\rho^2, A_3 = f(2c + f)^2\rho^2\phi^2, A_4 = 4c(c + f) - 4ad + f(f + 2d\rho^2), A_5 = 4c(c + f) - 4ad + f(f + 2d\phi^2), A_6 = 2c(c + f) + d(f\rho^2 - 2a), A_7 = 2c(c + f) + d(f\phi^2 - 2a), A_8 = e^{-z\xi(2\rho+\phi)} + e^{-z\xi\phi} - e^{-z\xi(\rho+2\phi)} - e^{-z\xi\rho}, A_9 = 1 - e^{-z\xi\rho}, A_{10} = e^{-z\xi\rho} + 1, A_{11} = 1 - e^{-z\xi\phi}, A_{12} = e^{-z\xi\phi} + 1, A_{13} = a(2d\phi^2 - f), A_{14} = a(2d\rho^2 - f), A_{15} = 2c + f, A_{16} = c + d\phi^2, A_{17} = c + d\rho^2, A_{18} = A_9A_{12}\phi(A_{13} - \rho^2A_2), A_{19} = A_{11}A_{12}\phi(\phi^2A_2 - A_{13}), A_{20} = A_{11}\rho A_{12}(\rho^2A_1 - A_{14}),$

$$A_{21} = A_{10}\rho(\phi^2A_1 - A_{14}), \quad A_{22} = (e^{-z\xi\phi}A_9\phi A_5 - A_{11}\rho A_4),$$

$$A_{23} = (e^{-z\xi\phi}A_9\phi A_1 - A_{11}e^{-z\xi\rho}\rho A_2), \quad A_{24} = (e^{-z\xi\phi}A_9\phi A_7 - e^{-z\xi\rho}A_{11}\rho A_6),$$

$$A_{25} = (e^{-z\xi\phi}A_9\phi A_1 - e^{-z\xi\rho}A_{11}\rho A_2), \quad A_{26} = (e^{-z\xi\rho}A_{12}A_{17} - e^{-z\xi\phi}A_{10}A_{16}),$$

$$A_{27} = (e^{-z\xi\rho}A_{12}A_4 - e^{-z\xi\phi}A_{10}A_5), \quad A_{28} = (e^{-z\xi\phi}A_9\phi A_1 - e^{-z\xi\rho}A_{11}\rho A_2),$$

$$A_{29} = df^2\xi\rho\phi.$$

(3) 当 $\beta^2 < 0$ 时, 令 $w^2 = -\beta^2$, 则 $k_{11} = \sqrt{a/d}\xi(-A_{12} + A_{11}fA_{13}A_{14} - A_{11}^2(A_{17} - 4A_2^2dA_6f\alpha w^2))/B_1,$

$$k_{12} = \xi(-a(A_1^2(A_{19} + A_{21}) - A_2^2dA_4^2A_{20}) + A_{23} + cA_{11}f^2A_{22})/B_1,$$

$$k_{13} = -4A_{11}e^{-z\alpha\xi}f\alpha\xi w(A_{11}f(A_1A_5\alpha + A_2A_4w) - 2adA_{18})/B_1,$$

$$k_{14} = 4aA_1dA_4e^{-z\alpha\xi}f(2c + f)\alpha\xi w/B_1, \quad k_{22} = -A_{11}\xi(A_1^2A_6A_7\alpha - A_{24} + A_{25} - A_{26} - A_{28})/B_1,$$

$$k_{24} = -4dA_{11}e^{-z\alpha\xi}f\alpha\xi w(A_1A_5(A_{11} - f)\alpha + A_2A_4(A_{11} + f)w)/B_1;$$

$$B_1 = -A_{11}f^2A_{29} + aA_4^2A_{30} - 4d(A_{31} - A_2^2A_4^2(A_{11} + f)w^2).$$

其中: $A_1 = \sin(z\xi w), A_2 = \cos(z\xi w), A_3 = \sin^2(z\xi w), A_4 = -e^{-2z\alpha\xi} + 1, A_5 = e^{-2z\alpha\xi} + 1, A_6 = -e^{-4z\alpha\xi} + 1, A_7 = 2A_{11}^2 - A_{11}^2f - A_{11}f^2 - c(c + f)(2A_{11} + f),$

$$A_8 = 2c^2 + 2cf + f(A_{11} + f), \quad A_9 = -A_{11} + f + 4d\alpha^2, \quad A_{10} = A_{11} - f + 4d\alpha^2, \quad A_{11} = \sqrt{ad}, \quad A_{12} = 2a^2A_1d^2A_4(A_1A_5\alpha - A_2A_4w),$$

$$A_{13} = (A_1A_5\alpha + A_2A_4w), \quad A_{14} = (A_1cA_4(c + f) + 2A_2dA_5f\alpha w),$$

$$A_{15} = f(A_9e^{-4z\alpha\xi} + 2A_{10}e^{-2z\alpha\xi} + A_9), \quad A_{16} = (2c^2A_4^2 + 2cA_4^2f + A_{15}),$$

$$A_{17} = -A_1^2A_6A_8\alpha + A_1A_2A_{16}w, \quad A_{18} = (A_1A_5\alpha - A_2A_4w),$$

$$A_{19} = 2c^3A_4^2 + c^2A_4^2f + 2cdA_5^2f\alpha^2, \quad A_{20} = f(-2c + 2A_{11} + f)w^2,$$

$$A_{21} = dA_5^2(2A_{11} - f)f\alpha^2, \quad A_{22} = (A_1^2A_5^2\alpha^2 - A_2^2A_4^2w^2),$$

$$A_{23} = A_{11}^2aA_1^2A_4^2(2c + f), \quad A_{24} = A_1A_2A_4^2c(2A_{11} - f)(c + f)w,$$

$$A_{25} = A_{11}^2A_4^2(2A_{11} + f)w, \quad A_{26} = 2A_2^2dA_6f(2A_{11} + f)\alpha w^2,$$

$$A_{27} = f((A_{11} - 2d\alpha^2) - 2e^{-2z\alpha\xi}(A_{11} + 2d\alpha^2)), \quad A_{28} = fA_1A_2(2 - A_6)$$

$$(4dA_{11}A_5^2\alpha^2 + A_{27})w, \quad A_{29} = (A_1^2A_5^2\alpha^2 - A_2^2A_4^2w^2),$$

$$A_{30} = (4c^2A_3 + 4cA_3f + A_3), \quad A_{31} = A_1^2A_5^2(A_{11} - f)\alpha^2.$$