A dynamic formulation of block element method
动力分析的块体单元法

LI Yong-ming(李永明), CHEN Sheng-hong(陈胜宏)
(School of Water Resources and Hydropower, Wuhan University, Wuhan 430072, China)

Abstract: The theories and applications of the hierarchical block element method are briefly introduced and a new formulation of the dynamic analysis is developed. Firstly, an overview of the development of the block element method is given. Secondly, the new concept of the covering element is explained and the static equilibrium equations are introduced. Using the shape functions of p-version finite element method, the displacement field of blocks are expressed as the functions of so called general degree of freedoms. Then the general stiffness matrix, mass matrix and damping matrix are listed in details, by which the general time-dependent inertia forces, damping forces and elastic forces distributing over the blocks are respectively transferred to the covering element nodes from in the blocks. And then the governing equations of the block dynamic system are deduced on the basis of the virtual work principle, the deformation compatibility condition and the constitutive relations. At last a numerical example is studied and the comparison between the calculated and analytical displacement response indicates the validity of the proposed method.

Key words: block element method; cover; dynamic analysis
CLC number: O 34; TU 457　　**Document code**: A　　**Article ID**: 1000-4548(2003)-05-0647-04

Biography: LI Yong-ming(1970-), male, doctor candidate of Wuhan University, engineer. Research field: stability, dynamic and crack analysis of rock slope and foundations.

摘　要: 介绍了阶谱块体单元法的基本理论和应用, 提出了动力分析的块体单元法表述公式。首先, 简要回顾了块体单元法的发展历程, 阐释了"覆盖"的新概念, 介绍了块体单元法的静力平衡方程组, 应用 p 型有限元法的形函数, 将块体的位移场表达成为所谓的广义自由度的函数. 然后, 给出了广义质量矩阵、广义刚度矩阵和广义阻尼矩阵的具体表达式, 通过它们将分布于块体内部的与时间有关的广义弹簧力、广义惯性力和广义阻尼力分别转移到覆盖单元的结点上. 随后, 基于虚功原理、变形协调条件和本构关系, 推导了块体动力系统的控制方程组. 最后, 研究了一个简单算例, 将计算结果和解析解进行了对比, 对比结果说明了所提出方法的正确性和有效性.

关键词: 块体单元法; 覆盖; 动力分析

0　Introduction*

The deformation and stability analysis of structures are inevitable in water resources and hydropower engineering. For this purpose, numerical computation is an important method besides the model test and engineering analogy. With the development of science and technology, it is more and more difficult to meet the needs of engineering practices for the classic numerical methods like limit equilibrium method and finite element method. Thus, a great number of new methods for solving the discontinuous media mechanical problems came forward in vast technical papers. In the field of rock engineering, many methods such as DEM[1], DDA[2] and NMM[3] etc. in which the rock masses are treated as discontinuous medias have been proposed during the last three decades. Besides, a distinctive method called the elastic-viscoplastic block element method by assuming that the rock blocks delimited by faults, joints and cracks etc. from the natural rock masses connect each other in a face-to-face way has also been put forward[4]. In the preliminary elastic-viscoplastic block element method, only the elastic-viscoplastic characteristics of discontinuities could be taken into account. The attempt to introduce the block deformation into account resulted in the presentation of a new elastic-viscoplastic block element method, in which the displacements of each block are described by the polynomial interpolation of the displacement field of the block[5]. But further study showed that the method performs well only when the order of the polynomial is lower than 3. The increase of the polynomial order will fail because of the poor numerical stability. Recently, a new solution procedure, which is called as the hierarchical block element method, was proposed[6]. In the new method, a conventional finite element is defined as the covering element for each block to comprise it. The displacements of the block are interpolated from the nodal displacements represented by the covering element. With the deformation compatibility condition of the blocks and the discontinuities, the elastic-viscoplastic constitutive relation together with the virtual work principle being taken into account, the governing equation can be established to solve the mapped displacements on the covering element. The coupled analysis method of the hierarchical block element method and the finite element method has been studied too[7]. Based on the above results, the preliminary study of the adaptive strategy of the hierarchical block element method based on the estimation of the error energy norm of each block element also has been studied[8]. The authors of the paper noted that the hierarchical block element method could also be used to deal with the mechanical analysis of continuum media structures by simply introducing some

* **Foundation item:** supported by NSFC(59979021)
Received date: 2002-11-22

virtual discontinuities with special material parameters into the continuous region[9]. Some important conclusions about the method also were drawn after the research on the order of shape functions, the geometric shape and density of block elements, as well as the material parameters on the virtual discontinuities.

As the theories and applications of both rigid and deformable block element method were all limited in the static analysis area, the authors try to develop a systematic methodology in this paper to extend the block element method into the dynamic analysis area. A numerical example is given to show the validity of the proposed method.

1 Basic theory of the hierarchical block element method

1.1 Notations

Usually, the variables in the nonlinear analysis are solved step by step. So in this paper, the relations among different static physical variables are given in the incremental form. Consider a block with the number of b_i. One of its neighbor blocks is denoted as the b_jth block, which connects it through the d_ith discontinuity. The displacement, strain, and stress increments of the b_ith block are expressed as $\{\Delta u\}_{bi}$, $\{\Delta\varepsilon\}_{bi}$, and $\{\Delta\sigma\}_{bi}$ while the deformation and stress increments of the d_ith discontinuity are denoted as $\{\Delta\delta\}_{di}$ and $\{\Delta\sigma\}_{di}$. The nodal displacement increments of the covering element of the b_ith block are denoted as $\{\Delta U\}_{bi}$. Accordingly, the velocity and the acceleration increments of the b_ith block will be denoted as $\{\Delta\dot{u}\}_{bi}$ and $\{\Delta\ddot{u}\}_{bi}$.

1.2 Concept of covering element

In the hierarchical block element method, the structures are assumed to be composed of the discontinuities and the blocks. Each block can be treated as one element. The block elements connect each other through the discontinuities lying among them. As it is difficult to describe the mechanical variables in the irregularly shaped block directly, a covering element is introduced for each block. The covering element is a conceptual minimum encircling cube of block (Fig. 1). In order to improve the accuracy, the displacement field of the covering element, which are represented by $\{\Delta U\}_{bi}$, are described by the p-version finite element hierarchical shape functions[10], and the true displacements of any point in the block element, denoted as $\{\Delta u\}_{bi}$, can be obtained by the interpolation approximation:

$$\{\Delta u\}_{bi} = [N]_{bi}\{\Delta U\}_{bi}(\text{in the } b_i\text{th block}), \quad (1)$$

where $[N]_{bi}$ is the matrix of the point, edge, face, or body shape functions as the case may be; $\{\Delta U\}_{bi}$ is the general displacement increments vector corresponding to the general degree of freedoms.

1.3 Static equilibrium equations

The loads acting on each block element are transferred to the respective nodal values of the covering element

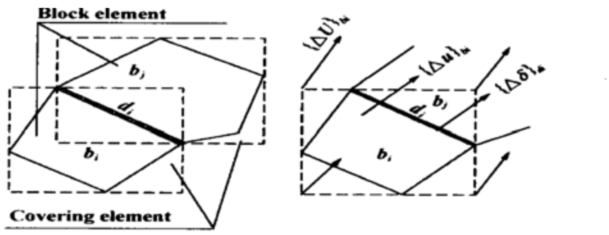

Fig. 1 Covering elements and their general displacement increments

$\{\Delta f\}_{bi}$ and the governing equilibrium equations can be established according to the virtual work principle as:

$$[K]\{\Delta U\} = \{\Delta F\}, \quad (2)$$

where $[K]$ is the global stiffness matrix whose elements are given by:

$$[K]_{bi,bi} = \iiint_{bi}[B]^{\mathrm{T}}_{bi}[D]_{bi}[B]_{bi}\mathrm{d}\,\Omega + \sum_{di,bi}\iint_{di}[N]^{\mathrm{T}}_{bi}[L]^{\mathrm{T}}_{di}[D]_{di}[L]_{di}[N]_{bi}\mathrm{d}S, \quad (3)$$

$$[K]_{bi,bj} = -\iint_{di}[N]^{\mathrm{T}}_{bi}[L]^{\mathrm{T}}_{di}[D]_{di}[L]_{di}[N]_{bj}\mathrm{d}S, \quad (4)$$

where $[B]_{bi}$ is the strain matrix; $[L]_{di}$ is the transformation matrix, which is the function of the dip direction and dip angle of the d_ith discontinuity; $[D]_{bi}$ and $[D]_{di}$ are the elastic matrices of the b_ith block and the d_ith discontinuity respectively. $[D]_{bi}$ is the function of the Young's modulus and Poisson's ratio and $[D]_{di}$ is the function of the normal and tangential stiffness coefficients; $\{\Delta U\}$ is the global general displacement increments vector; and $\{\Delta F\}$ is the global general load increments vector:

$$\{\Delta F\} = \{\Delta F_{b1}, \Delta F_{b2}, \cdots, \Delta F_{bn}\}, \quad (5)$$

where the subscript bn is the total number of blocks, while

$$\{\Delta F\}_{bi} = \{\Delta f\}_{bi} + \{\Delta F^{vp}\}_{bi} \quad (6)$$

and

$$\{\Delta f\}_{bi} = [N]^{\mathrm{T}}_{bi}\Big|_{(x_q,y_q,z_q)}\{\Delta q\} + \iint_{di}[N]^{\mathrm{T}}_{bi}\{\Delta p\}\mathrm{d}S + \iiint_{bi}[N]^{\mathrm{T}}_{bi}\{\Delta v\}\mathrm{d}\,\Omega \quad (7)$$

is the general external load increments vector, which can be obtained by load transferring based on the virtual work principle. $\{\Delta q\}$ is the point load increments vector acting at the (x_q, y_q, z_q) point; $\{\Delta p\}$ is the surface load increments vector and $\{\Delta v\}$ is the volume load increments vector. The 2nd term in the right hand in Eq. (6) can be represented as:

$$\{\Delta f^{vp}\}_{bi} = -\iiint_{bi}[B]^{\mathrm{T}}_{bi}\{\Delta\sigma^{vp}\}_{bi}\mathrm{d}\,\Omega - \sum_{di,bi}\iint_{di}[N]^{\mathrm{T}}_{bi}[L]^{\mathrm{T}}_{di}\{\Delta\sigma^{vp}\}_{di}\mathrm{d}S. \quad (8)$$

They are the equivalent load increments of the viscoplastic deformation of both blocks and discontinuities, $\{\Delta\sigma^{vp}\}_{bi}$ and $\{\Delta\sigma^{vp}\}_{di}$ are the viscoplastic stress increments of the b_ith block and the d_ith discontinuity. Both of them can be given according to the flow law while different yielding criteria are adopted for the block and the discontinuity. The subscript d_i runs over the discontinuities around the b_ith block, and the subscript b_j runs over the neighbor blocks around the b_ith block corresponding to the d_ith discontinuity.

With the solved general nodal displacement increments $\{\Delta U\}_{bi}$ of the covering element, the displacement, the strain as well as the stress increments in the block b_i can be calculated.

2 Formulation of the dynamic analysis

The formulation and the procedure are similar to the finite element dynamic analysis. Transferring the general time-dependent inertia forces, damping forces and elastic forces distributing over the block from in the block to the covering element nodes by the general mass matrix, damping matrix and stiffness matrix respectively, the governing equations of the block dynamic system can be deduced on the basis of the virtual work principle. These forces are unknown but they can be expressed by the displacement, velocity and acceleration.

Although each block is originally assumed to be continuum, it is treated as a concentrated mass system of multi-degree-of-freedom in computation.

Based on the face-connection assumption, the basic equations can be established after the formation of the equilibrium equations of the block system, the geometry compatibility equations of the deformations of blocks and discontinuities and the displacements of blocks, as well as the elastic-viscoplastic constitutive equations of the deformations and stresses of blocks and discontinuities.

2.1 General stiffness matrix of the block element

The general stiffness matrix has been interpreted above in the static governing equations. But if the direct integration method is employed to solve the nonlinear motion equations step by step, the effect of mass and damp should be introduced to form a general effective stiffness matrix, which will be discussed in detail later in this paper.

As the stiffness matrix should match the general degree-of-freedom, it is called as general stiffness matrix. For the same reason, the mass matrix and the damping matrix are general matrices.

2.2 General mass matrix of the block element

Denote the mass density of the block element b_i as ρ_{bi}, and set the inertia forces at any point in the block equal the equivalent inertia forces on the nodes of the covering element, then the general compatible mass matrix can be obtained:

$$[M]_{bi} = \iiint_{\Omega_{bi}} [N]^{\mathrm{T}}_{bi}\rho_{bi}[N]_{bi}\mathrm{d}\,\Omega. \qquad (9)$$

However, for the sake of convenience, the mass matrix can also be formed by simply distributing the block mass to the nodes of the covering element, which is called as concentrated mass matrix and it becomes diagonal:

$$[M]_{bi} = \mathrm{diag}(m_1, m_2, \cdots, m_n) \qquad (10)$$

2.3 General damping matrix of the block element

The magnitude of damp plays a decisive roll to the response of the structure in the case of forced vibration. In order to decouple the damping forces and the vibration mode arise from them, the Rayleigh damping matrix is adopted. The Rayleigh damping matrix of block element b_i can be expressed as:

$$[C]_{bi} = \alpha[M]_{bi} + \beta[K]_{bi}, \qquad (11)$$

where α, β are the damping factors need to be evaluated from the test.

2.4 Governing equations of dynamic analysis

The dynamic response of a MDOF system is governed by the following equations of motion:

$$[M]\{\Delta\ddot{u}\} + [C]\{\Delta\dot{u}\} + [K]\{\Delta u\} = \{\Delta R\}, \qquad (12)$$

where $\{\Delta R\}$ is the load increments vector of time-dependent load functions; $[M]$, $[C]$ and $[K]$ are the general mass matrix, general damping matrix and general stiffness matrix matching the order of the general displacement increments vector $\{\Delta u\}$. They are all symmetry square matrices. After solve the motion equations the covering element nodal general displacement response of every block can be obtained and then the general velocity and general acceleration response can also be calculated. Note that if the system is undamped, the damping term $[C]\{\Delta\dot{u}\}$ will be absent from the equations of motion. And also note that if the Wilson-θ method is adopted to solve the motion equations step by step, the general displacement increments at any discrete time station need to be solved indirectly.

2.5 Solver of the dynamic motion equations

The dynamic response governing equations(12) are a set of nonlinear equations. The step-by-step integration method can be employed to solve them. The average acceleration method, the linear acceleration method, the Newmark β method, the Wilson-θ method, and so on are the direct integration methods often been employed. Since the Wilson-θ method is stable at any length of time interval when the value of the parameter θ is properly chosen[11] so it will be adopted in this paper. In this method, the accelerations vary linearly in the time interval t to $t+\theta\Delta t$. As a primary study, only the linear displacement and the elastic constitutive relation are taken into account and the dynamic equilibrium equations can be rewritten as:

$$[\bar{K}]\{U_{t+\theta\Delta t}\} = \{\bar{R}_{t+\theta\Delta t}\}, \qquad (13)$$

where $[\bar{K}]$ is the effective stiffness matrix:

$$[\bar{K}] = [K] + \frac{3}{\theta\Delta t}[C] + \frac{6}{\theta^2\Delta t^2}[M], \qquad (14)$$

$\{\bar{R}_{t+\theta\Delta t}\}$ is the effective load vector at time station $t+\theta\Delta t$:

$$\begin{aligned}\{\bar{R}_{t+\theta\Delta t}\} = {} & \{R_t\} + \theta(R_{t+\Delta t} - R_t) + \\ & [M]\left|2\{\ddot{U}_t\} + \frac{6}{\theta\Delta t}\{\dot{U}_t\} + \frac{6}{\theta^2\Delta t^2}\{U_t\}\right| + \\ & [C]\left|\frac{\theta\Delta t}{2}\{\ddot{U}_t\} + 2\{\dot{U}_t\} + \frac{6}{\theta\Delta t}\right|\{U_t\}.\end{aligned} \qquad (15)$$

After the displacement at $t+\theta\Delta t$ is worked out, the acceleration, velocity and displacement at $t+\Delta t$ can also be calculated.

3 Example study

Consider a left-end-fixed elastic bar shown in Fig. 2. There exists a concentrated tension load P_0 acting on the right end before it is released at time station $t=0$. The longitudinal undamped vibration of the bar corresponding to the 1st mode will be studied.

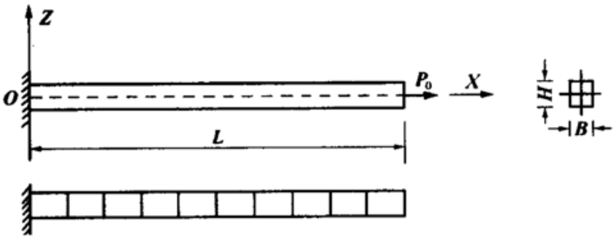

Fig. 2 The left-end-fixed bar and the meshes

The initial conditions are:

$$u(x,0)=\frac{P_0x}{EA},\ \frac{\partial u}{\partial t}\bigg|_{(x,0)}=0, \qquad (16)$$

the analytical solutions are given as[12]:

$$u(x,t)=\sum_{k=1}^{\infty}B_k\sin(G_kx)\cos(G_kC_Rt), \qquad (17)$$

where

$$\left.\begin{aligned}B_k&=\frac{8u_0(-1)^{k+1}}{(2k-1)^2\pi^2} \quad (k=1,2,\cdots)\\ G_k&=\frac{(2k-1)\pi}{2L} \quad (k=1,2,\cdots)\end{aligned}\right|, \qquad (18)$$

and $u_0=u(L,0)$ is the initial displacement at the right end; $C_R=\sqrt{E/\rho}$ is the longitudinal wave propagation velocity.

The bar is a continuous structure and some virtual discontinuities need to be introduced to divide the bar into elements in the block element method. Here altogether 10 elements are divided and only elastic constitution relations are considered.

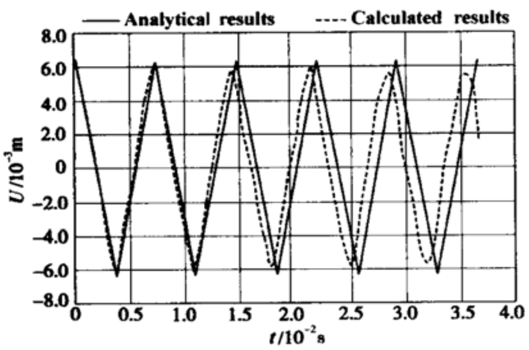

Fig. 3 The comparison between the calculated and analytical displacement response

Fig. 3 is the comparison between the calculated and the analytical displacement response. From the comparison the conclusion can be drawn that the method proposed in this paper is effective for the dynamic analysis of structures.

4 Conclusions

A new p-version block element method is developed which can be used to analyze both the static deformation and dynamic response of continuum media structures and discontinuous structures. A simple example shows that the method can give the result in good agreement with the analytical solution. However, there are some important problems such as the error estimation and the adaptation of time step length to be solved before the method can be used in the complicated rock engineering structures.

References:

[1] Cundall P A. Formulation of a three-dimensional distinct element model[J]. Int J Rock Mech Min Sci & Geomech, 1988, **25**(3): 107-125.

[2] Shi G H. Discontinuous deformation analysis: a new numerical model for the statics and dynamics of deformable block structures [J]. Eng Comput, 1992, **9**(2): 157-168.

[3] Shi G H. Numerical manifold method and discontinuous deformation analysis[M]. Pei J M Translated. Beijing: TsingHua University Press, 1997.

[4] Chen S H, Wang H R, Xiong W L. A Preliminary study on three dimensional elastic-viscoplastic block theory for rock masses [J]. J of Wuhan Univ of Hydr & Electr Eng, 1990, **23**(6): 55-61.

[5] Chen S H. A new development in elastic-viscoplastic block theory of rock masses[A]. In: Proc 8th Int Conf Computer Meth & Adv. In Geomech[C]. West Virginia: A A Baalkema, 1994. 2235-2240.

[6] Wang W M, Chen S H. A new implementation of elastic-viscoplastic block theory for rock masses[A]. Kundu T, Desai C S, eds. Proc 10th Int Conf on Compu Meth & Adv in Geomech [C]. Arizona: A A Balkema, 2001. 417-422.

[7] Wang W M, Xu M Y, Chen S H. Coupled method of block element and finite element for hydraulic structures[J]. Chinese Journal of Rock Mechanics and Engineering, 2001, **20**: 1029-1033.

[8] Wang W M. Theories and practices of the three-dimensional elastic-viscoplastic block element method for rock masses[D]. Wuhan: Wuhan University, 2001.

[9] Li Y M, Wang W M, Chen S H. Hierarchical block element method and its application to continuum mechanical problems [J]. Rock and Soil Mechanics, 2002, **23**(3): 362-367.

[10] Cheng Z, Chen S H. 3-D hierarchical FEM analysis for hydraulic structure[J]. J Hydraulic Engineering, 1999, (12): 53-58.

[11] Bathe K J. Finite element procedures in engineering analysis [M]. Englewood Cliffs, New Jersey: Prentice-Hall, 1982.

[12] Wang W L, et al. Dynamics of structures[M]. Shanghai: Fudan University Press, 1993.