An analytical approach to one-dimensional finite strain non-linear consolidation by Lie group transformation
李群变换求解一维非线性有限变形固结问题

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Abstract: Based on some assumptions for the relationship between permeability and void ratio and between effective stress and void ratio of the clay, the non-linear equation governing the progress of one-dimensional consolidation of the saturated clay, namely Gibson's Equation, is discussed under different initial and boundary conditions. The properties of different kinds of transformed boundaries are evaluated. It is shown that for some relatively simple conditions, exact analytical solutions of the non-linear consolidation equation can be obtained by using the approach of the transformation of Lie group. Finally, the solution obtained in this paper is compared with Tan & Scott's solutions, and some problems that appeared in solutions are analyzed.

Key words: nonlinear; finite strain; consolidation; analytical solution; Lie group; settlement
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摘 要: 在对渗透系数与孔隙比以及有效应力与孔隙比的关系作出假设的基础上,讨论了不同初值和边值条件下的饱和粘土一维非线性有限变形固结方程,同时对不同边界条件的性质进行了评价。研究表明,对于一些相对简单的边界条件,通过李群变换的方法可以得到非线性固结方程的完整解析解。最后,将所得到的解答与Tan & Scott的有关解答进行了分析比较,并探讨了求解过程中出现的一些问题。

关键词: 非线性; 有限变形; 固结; 解析解; 李群; 沉降

1 Introduction[y]

On the basis of the assumptions more general than those usually adopted by Terzaghi, Biot, etc., Gibson et al derived a differential equation governing the progress of one dimensional consolidation of saturated clays[1,2]. In terms of reduced coordinates (z, t), the equation can be shown as a nonlinear form:

$$\pm\left(\frac{\gamma_s}{\gamma_f}-1\right)\frac{\mathrm{d}}{\mathrm{d}e}\left(\frac{k(e)}{1+e}\right)\frac{\partial e}{\partial z}+\frac{\partial}{\partial z}\left[\left(\frac{1}{\gamma_f}\frac{k(e)}{1+e}\frac{\mathrm{d}\sigma'}{\mathrm{d}e}\right)\frac{\partial e}{\partial z}\right]+\frac{\partial e}{\partial t}=0 \tag{1}$$

where γ_f and γ_s are the specific gravity of the fluid and the skeleton of soil respectively; e is the void ratio; $k(e)$ is the coefficient of permeability, which is a function of void ratio; σ' is the effective stress defined by Terzaghi. The operator "$\pm$" means that the z-axis is against/with gravity. Concentrated research was done on Eq.(1) during the past decades with the assumption that parameters $g(e)$ and $\lambda(e)$ expressed by following equations respectively are constant.

$$g(e)=-\frac{1}{\gamma_f}\frac{k(e)}{1+e}\frac{\mathrm{d}\sigma'}{\mathrm{d}e} \tag{2}$$

$$\lambda(e)=\frac{\mathrm{d}}{\mathrm{d}e}\left(\frac{\mathrm{d}e}{\mathrm{d}\sigma'}\right) \tag{3}$$

But Duncan pointed out that it was not reasonable to suppose constant $g(e)$ and $\lambda(e)$[3]. Xie Xinyu et al had shown that the relationship between the void ratio and the effective stress of soils is not linear even if the strain-stress relation is linear in finite strain theory[4]. Eq.(1) can also be transformed into equations of other dependent variables, such as porosity[5], natural strain and compression ratio[6], and displacement by using Lagrangian or Eulerian or convective coordinates[4].

Many experiments show that the relationship between $k(e)$ and e, can be expressed approximately as

$$e=e_0+A(\ln k-\ln k_0) \tag{4}$$

namely

$$k=k_0\exp\left(\frac{(e-e_0)}{A}\right)=m\exp\left(\frac{e}{A}\right) \tag{5}$$

where

$$m=k_0\exp\left(-\frac{e_0}{A}\right) \tag{6}$$

The compression texts show that, for the unique loading and unloading, the relationship between σ' and e can be expressed approximately as

$$e=e_0-C(\ln\sigma'-\ln\sigma_0') \tag{7}$$

or

$$\sigma'=\sigma_0'\exp\left(-\frac{e-e_0}{C}\right)=n\exp\left(-\frac{e}{C}\right) \tag{8}$$

where

$$n=\sigma_0'\exp(e_0/C) \tag{9}$$

With Eqs.(5) and (8) being considered, Eq.(1) can

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be written as

$$\pm\left[\left(\frac{\gamma_s}{\gamma_f}-1\right)\frac{d}{dv}\left[\frac{m\exp(v/A)}{1+v}\right]\frac{\partial v}{\partial z}\right]-\frac{\partial}{\partial z}\left[\left(\frac{m}{\gamma_f}\frac{\exp(v/A)}{1+v}\frac{n}{C}\exp\left[\frac{-v}{C}\right]\right)\frac{\partial v}{\partial z}\right]+\frac{\partial v}{\partial t}=0 \qquad (10)$$

As is known to us, it is difficult to obtain an analytical solution, even an approximate one for this non-linear equation without any further presumption.

On the other hand, the feasibility of using Eq. (7) to express the relationship between void ratio and effective stress during the whole stress history is questionable, since the parameter C in Eq. (7) may not be constant under different conditions of consolidation of soils.

2 Simplification of the theory

Taking the above-mentioned reasons into consideration, we can fit the data from one-dimensional compression tests with a hyperbolic curve:

$$\sigma' = \frac{a(e_0 - e)}{e - b} \qquad (11)$$

where a and b are regression constants depending on the data of tests; e_0 is the void ratio while effective stress is null.

Monte et al[7] obtained a relationship between the coefficient of permeability and the void ratio on the basis of Kozeny-Carman Equation (Carman, 1959), which appeared in the gas flowing through the porous medium.

$$k = \frac{\gamma_f}{\mu_f}\frac{T}{C_p S^2}\frac{e^3}{(1+e)} \qquad (12)$$

where γ_f and μ_f are the specific gravity and the viscosity of fluid respectively; C_p is the pore shape factor; S is the specific surface per unit volume of particles; and T is the tortuosity. The relationship between k and e obtained by Monte et al can be expressed as[7]

$$k(e) = (1+e)(c_1 + c_2 e) \qquad (13)$$

where c_1 and c_2 are constants for each increment of loading. But their experiments showed subsequently that the ratio of the tortuosity to the shape factor decreased as the void ratio decreased. Meanwhile, their experiments showed that parameter A in Eq. (4) is not a constant throughout the entire range of the investigated void ratio. So we can adopt the relationship of $k(e) - e$ in a short range of the void ratio as a linear form:

$$k(e) = m(1+e) \qquad (14)$$

where m can be evaluated by the permeability test. By using these presumptions, Eq. (1) can be rewritten as

$$\frac{\partial}{\partial z}\left[\left(\frac{1}{\gamma_f}m\frac{a(e_0-b)}{(e-b)^2}\right)\frac{\partial e}{\partial z}\right] = \frac{\partial e}{\partial t} \qquad (15)$$

and can be further simplified to

$$-\lambda\frac{\partial}{\partial z}\left[\frac{1}{(e-b)^2}\frac{\partial e}{\partial z}\right]+\frac{\partial e}{\partial t} = 0 \qquad (16)$$

where λ is a constant, ie.

$$\lambda = \frac{m}{\gamma_f}a(e_0 - b) \qquad (17)$$

3 Well-posed conditions

The well-posed conditions of the problem, mentioned previously, can be expressed as follows.

(1) PTIB condition (pervious top and impervious bottom)

For the conditions of PTIB, the initial value condition that the soil layer is thin or thick can be written as following equations respectively:

$$e(z,0) = e_0 \qquad (18)$$

or

$$e(z,0) = a_1 + a_2 z^2 \qquad (19)$$

The boundary conditions of the top and the bottom of soil layer can be written respectively as

$$e(0,t) = e_t \qquad (20)$$

and

$$\left.\frac{\partial e}{\partial z}\right|_{z=h} = -\left.\frac{\gamma_s-\gamma_f}{(1+e_0)\,d\sigma'/de}\right|_{z=h} \qquad (21)$$

Combined with Eq. (11), Eq. (21) can be expressed as

$$\left.\frac{\partial e}{\partial z}\right|_{z=h} = -\frac{\gamma_s-\gamma_f}{(1+e_0)}\Bigg/\left.\left[\frac{a(e_0-b)}{(e-b)^2}\right]\right|_{z=h} \qquad (22)$$

(2) PTPB condition (pervious top and pervious bottom)

For the conditions of PTPB, its initial condition and boundary condition of the top of soil layer are the same as that of PTIB. The boundary condition of the bottom of soil layer is the same as that of the top, namely,

$$e(0,t) = e_b \qquad (23)$$

4 The exact analytical solutions

Generally, let

$$e - b = v + 1$$
$$\tau = \lambda t$$

Eq. (16) becomes

$$-\frac{\partial}{\partial z}\left[\frac{1}{(v+1)^2}\frac{\partial v}{\partial z}\right]+\frac{\partial v}{\partial \tau} = 0 \qquad (24)$$

By using the approach of Lie group transformation[8], Eq. (24) can be changed into a linear parabolic form:

$$-\frac{\partial^2 \bar{v}}{\partial \bar{z}^2}+\frac{\partial \bar{v}}{\partial \bar{\tau}} = 0 \qquad (25)$$

By admitting the following transformation

$$\left.\begin{aligned}\bar{\tau} &= \tau - \tau_0\\ \bar{z} &= \int_{z_0}^{z} v\,dz' - \int_{\tau_0}^{\tau}\left[\frac{\partial}{\partial z}v^{-1}\right]_{z=z_0} d\tau'\\ \bar{v} &= v^{-1}\end{aligned}\right\} \qquad (26)$$

where (z_0, τ_0) is an arbitrary fixed point. We can easily get

$$\left.\begin{aligned} \tau &= \bar{\tau} - \bar{\tau}_0 \\ z &= \int_{\bar{z}_0}^{\bar{z}} \bar{v}\,\mathrm{d}z' - \int_{\bar{\tau}_0}^{\bar{\tau}} \left| \frac{\partial}{\partial \bar{z}} \bar{v}^{-1} \right|_{\bar{z}=\bar{z}_0} \mathrm{d}\bar{\tau} \\ v &= \bar{v}^{-1} \end{aligned}\right| \tag{27}$$

And the solution of Eq. (24) would be

$$v(z, \tau) = \frac{1}{\bar{v}(\bar{z}(z, \tau), \bar{\tau})} \tag{28}$$

For the special case of the well-posed condition on the semi-infinite homogenous space, the problem can be solved analytically by applying a similar transformation, namely, the Boltzmann transformation[9].

Here we choose a relatively simple but important example, which is in a semi-infinite one-dimensional space. The governing equation can be expressed as Eq. (15), whose initial condition is

$$e(z, 0) = e_0 \tag{29}$$

And boundary value conditions for PTPB are

$$e(0, t) = e_f \tag{30}$$

$$e(\infty, t) = e_0 \tag{31}$$

It can be treated as a homogeneous thick layer with initial void ratio e_0, whose top and bottom boundaries are both pervious. And when loading is applied, the initial void ratio e_0 will become the final void ratio e_f in the top of the layer. With these conditions of Eqs. (23) and (29) ~ (31), the exact solution of Eq. (15) can be obtained by the method mentioned above.

$$\frac{e_0 - e}{e - b} = \frac{e_0 - b}{e_f - b}\sqrt{\pi}\alpha \cdot \exp(\alpha^2) \cdot [1 - \mathrm{erf}(\alpha\beta)] \tag{32}$$

where erf is the error function; α and β are the parameters depending on the following two equations.

$$\sqrt{\pi}\alpha \cdot \exp(\alpha^2) \cdot [1 - \mathrm{erf}(\alpha)] = \frac{e_0 - e_f}{e_0 - b} \tag{33}$$

and

$$\frac{z}{\sqrt{\lambda}} = \frac{-2\alpha}{e_f - b}\exp[\alpha^2(1 - \beta^2)] + 2\alpha\beta(e_f - b) \tag{34}$$

Finally, we can obtain the equation of settlement of the top boundary by laborious calculation:

$$s(t) = \int_0^{\infty} (e_0 - e)\,\mathrm{d}z = 2\alpha\sqrt{\lambda} \tag{35}$$

5 Compared with Tan & Scott's solutions

Tan & Scott put forward analytical solutions for one-dimensional consolidation of a semi-infinite layer under uniform initial condition with or without the convective effect[10]. To simplify the problems, a coefficient of consolidation defined as

$$c_v = \frac{k(\varepsilon)}{\gamma_f}\frac{\mathrm{d}\sigma'}{\mathrm{d}\varepsilon} \tag{36}$$

is assumed to be constant, where ε is natural strain defined as

$$\varepsilon = \ln\left|\frac{1 + e_0}{1 + e}\right| \tag{37}$$

The solutions to the finite strain consolidation of a semi-infinite layer with or without the convective effect were given as

$$s(t) = 2p\sqrt{c_v t} \tag{38}$$

and

$$s(t) = q\sqrt{c_v t} \tag{39}$$

respectively, where p and q are the solutions to the following two equations respectively:

$$\sqrt{\pi}p\exp(p^2)[1 - \mathrm{erf}(p)] = \frac{e_0 - e_f}{1 + e_0} \tag{40}$$

$$q\exp\left(\frac{q^2}{4}\right)\left[1 - \mathrm{erf}\left(\frac{q}{2}\right)\right] = \frac{2\varepsilon_f}{\sqrt{\pi}} \tag{41}$$

It was found that when the degree of consolidation U is not greater than 52.6%, there exist the direct proportional relationship between U and the square root of time factor $\sqrt{T_v}$ based on the analytical solution of Terzaghi's one-dimensional consolidation theory. Furthermore, there may exist the direct proportional relationship between settlement s and the square root of time $\sqrt{t}$ when the degree of consolidation is small.

From Eqs. (35), (38) and (39), it can be seen that these expressions for settlement are quite similar. They all show the direct proportional relationship between settlement and the square root of time. The differences between the solution obtained in this paper and Tan & Scott's solutions, mainly rest in the parameters α, p and q, as well as λ and c_v. The latter results from the various assumptions of the relationship between permeability and void ratio and the one between effective stress and void ratio.

The assumption of the relationship between the coefficient of permeability and the void ratio, i. e. Eqs. (13) and (14) implicitly means that the soil is permeable while the void ratio is zero. This is apparently unacceptable until the void ratio of soils becomes smaller. But the void ratio in the engineering practice can not become too small. The experiments by Xue Xingdu et al also testified it[11]. In the process of theoretical calculation, we find out that the parameters in the relationship between the void ratio and the effective stress sensibly depend on the smallest void ratio from the compression test. It is the result that the numerical program in the non-linear regression is applied[12].

For the different well-posed conditions of the problem, the solution should be iterated dissimilarly from which different forms of the solutions would be obtained. Here we do not refer to an explicit analytical solution about the void ratio depended on the variables of the depth and time. The solution for more general conditions (such as for the clay layer with finite thickness) will be discussed in a later paper.

6 Conclusions

The Gibson's Equation for some soils can be simplified, and its analytical solution, although it is non-linear, can be obtained under more complex conditions than before. Compared with Tan & Scott's solutions, the results show that the simplification mentioned above is reasonable.

For the saturated thick clay layer, the model used here can be applied to evaluate its settlement approximately.

The consolidation settlement of soils has a linear relationship with the square root of time in the process of primary consolidation. The parameters mainly depend on the data from the compression test.

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