A one-dimensional geosynthetic-reinforced foundation model with non-linear spring support
土工布加筋基础的一维非线性模型

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Abstract This paper presents a one-dimensional geosynthetic-reinforced foundation model with hyperbolic spring support representing non-linear soft soil reaction. The mathematical expressian of this model is a set of two non-linear ordinary differential equations. A finite difference iteration procedure is suggested for solving these two differential equations. This paper focuses on the influence of one normalised parameter in the normalised hyperbolic equation for springs on the settlement and mobilised tension force in the geosynthetics. The suggested model may be used as a basis for considering more complexities such as elasto-plastic behaviour of sand and elasto-plastic interface behaviour between the geosynthetics and the sand.

Keywords geosynthetics，foundation，settlement，soft soil，deformation compatibility，non-linear springs，finite difference.

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文 摘 本文推导出一个代表软土双曲线弹簧支撑的土工布加筋基础一维非线性模型。该模型的数学形式是两个非线性二阶常微分方程组。给出了解常微分方程组的迭代格式，并着重讨论了非线性弹簧参数对加筋基础的沉降和土工布拉力的影响。

关键词 土工布，基础，沉降，变形协调，非线性弹簧，有限差分。
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1 Introduction

Foundations on soft soils may have problems such as excessive settlement and bearing capacity failure. One of techniques for improvement is to place sand/gravel fills with one or multiple layers of geosynthetics on soft foundation soils. There is a question of how to calculate the settlement of sand/gravel fills and the tension in the geosynthetics. A lot of research work in this area has been done，for example，Shukla and Chandra[1]. Their models simplify the two-dimensional（2-D）plane strain problem（e. g. strip footing）into a one-dimensional（1-D）problem. Soft soils are represented by linear or non-linear springs. The sand fills on or below a geosynthetic layer（geomembrane，geogrid，or equivalent）are assumed to behave like a Pasternak shear layer[2]. This type of models has one problem which is the same as that in the Winkler foundation models. This problem is that the interaction between springs is ignored. Shukla and Chandra（1995）pointed out that for soft soils，the interaction is not strong. The 1-D models consider the main characteristics of geosynthetic-reinforced foundation soils and the computation of the model problems is fast. Therefore，this 1-D model approach has its value in application.

Yin（1997）proposed a 1-D model which can consider the deformation compatibility between soil and geomembrane. However，in Yin´s model[3]，soft soils are represented by liner springs. This paper extends Yin´s model（1997）by using non-linear springs to represent soft soils. *

2 One-dimensional foundation model

Fig. 1 shows a 1-D foundation model of geosynthetic-reinforced sand fills over soft soils（non-linear springs）. Fig. 2 shows three elements（infinitesimal in horizontal direction）and forces（positive directions as shown）. Following a common model development approach from simple to complicated as used by others（Shukla and Chandra 1995），the behaviour of the sand fills is assumed to be linear elastic and there is no slip between sand fills and geosynthetics. The model in this paper can be extended for considering slip.

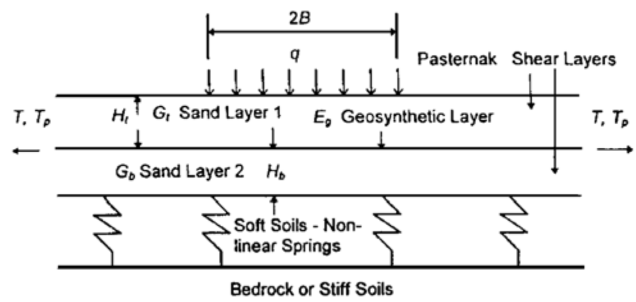

Fig. 1 1-D foundation model with non-linear spring support

Considering the vertical force equilibrium of element 1 in Fig. 2 leads to

$$q\mathrm{d}x - \sigma_n \mathrm{d}l\cos\theta + \tau_n \mathrm{d}l\sin\theta + \frac{\mathrm{d}\tau}{\mathrm{d}x}\mathrm{d}xH_t = 0 \tag{1}$$

where q = pressure from footing; $\mathrm{d}x$ = infinitesimal element length（horizontal）; σ_n and τ_n = the normal and shear stresses at the interface of sand/geosynthetics; $\mathrm{d}l$ and θ = infinitesimal element length（stretched）and rotation angle; H_t = element thickness; τ = average element shear stress（on vertical side）. Since the sand behaviour is assumed to be elastic and the settlement computed is the part caused by the footing pressure q，the self weight of the sand fills may not be required to be considered here. If required，the pressure due to the self weight may be included in the loading pressure q.

According to the Pasternak shear layer assumption，the following relationships exist:

$$\tau = G_t\frac{\mathrm{d}w}{\mathrm{d}x} \qquad \frac{\mathrm{d}\tau}{\mathrm{d}x} = G_t\frac{\mathrm{d}^2w}{\mathrm{d}x^2} \tag{2}$$

where G_t = the upper sand layer´s average shear modulus; w = vertical settlement. Using Equation（2），Equation（1）becomes

$$q = \sigma_n - \tau_n\tan\theta - H_tG_t\frac{\mathrm{d}^2w}{\mathrm{d}x^2} \tag{3}$$

Using the same approach and considering the vertical force equilibrium of element 3 result in:

$$q_s = \sigma_n{}' - \sigma_n{}'\tan\theta - H_bG_b\frac{\mathrm{d}^2w}{\mathrm{d}x^2} \tag{4}$$

where q_s = reaction pressure from the spring（soft soil）; $\sigma_n{}'$ and $\tau_n{}'$ = normal and shear stresses between geosynthetics and sand; H_b = thickness of the lower sand fill; G_b = the lower sand layer' s average shear modulus.

* Received March 27，1997.

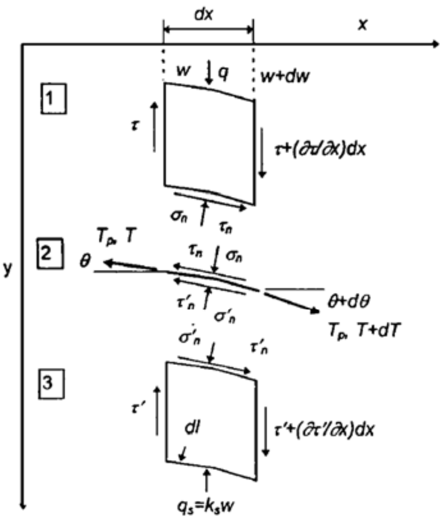

Fig. 2 Three elements and forces (after Yin 1997).

Taking the vertical force equilibrium of element 2

$$\frac{\mathrm{d}T}{\mathrm{d}x}\sin\theta = (T + T_p)\cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}x} - (\sigma_n - \sigma_n{}') + (\tau_n + \tau_n{}')\tan\theta \tag{5}$$

where T = the tension in geosynthetic layer; T_p = pre-tension.

Considering the horizontal force equilibrium of element 2

$$\frac{\mathrm{d}T}{\mathrm{d}x}\cos\theta = (T + T_p)\sin\theta\frac{\mathrm{d}\theta}{\mathrm{d}x} + (\sigma_n - \sigma_n{}')\tan\theta + (\tau_n + \tau_n{}') \tag{6}$$

In the derivation of Equation (6), it is used that $\sin(\theta + \mathrm{d}\theta) = \sin\theta\cos\theta(\mathrm{d}\theta) + \cos\theta\sin(\mathrm{d}\theta) \approx \sin\theta + \cos(\mathrm{d}\theta)$ (sice $\mathrm{d}\theta$ is small) and $\mathrm{d}l\cos\theta = \mathrm{d}x$.

If there is no slip between sand fill and geosynthetic layer, the following deformation compatibility conditions (relationships) exist (see Fig. 3):

$$u_x = u_x{}' = u_{g,x} \tag{7}$$

where u_x, $u_x{}'$, $u_{g,x}$ = horizontal displacement of the upper and the lower sand layers and the geosynthetic layer at the interface.

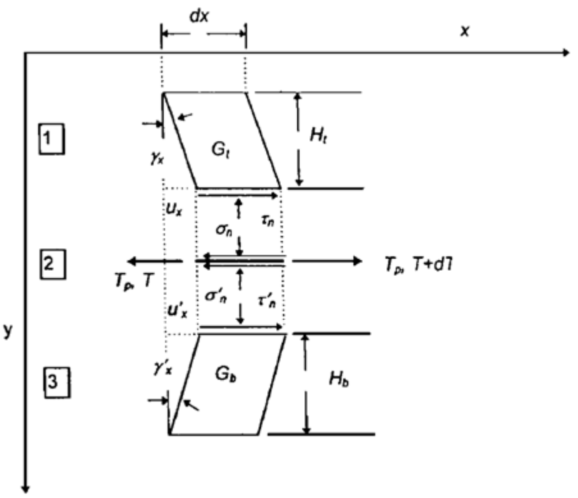

Fig. 3 Deformation comparability between sand and geosynthetics (after Yin 1997)

The upper and lower sand layers' displacements can be written as

$$u_x = H_t\gamma_x = H_t\frac{\tau_n}{G_t}$$
$$u_x{}' = H_b\gamma_x{}' = H_b\frac{\tau_n{}'}{G_b} \tag{8}$$

Using Equations (7) and (8), the shear stress is

$$\tau_n = \frac{G_t}{G_b}\frac{H_b}{H_t}\tau_n{}' \tag{9}$$

Fig. 4 shows a stretched and rotated geosynthetic element 2. Using Equations (7) and (8), the geosynthetic displacement and its differentiation can be written as

$$u_{g,x} = u_x{}' = \frac{H_b}{G_b}\tau_n{}' \tag{10}$$
$$\mathrm{d}u_{g,x} = \mathrm{d}u_x{}' = \frac{H_b}{G_b}\mathrm{d}\tau_n{}'$$

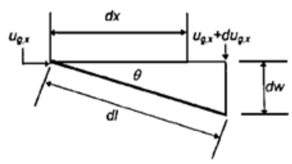

Fig. 4 Stretching and rotation of geosynthetic element

The geosynthetic element length $\mathrm{d}l$ after stretching and rotation is

$$\mathrm{d}l = \sqrt{(\mathrm{d}w)^2 + (\mathrm{d}l' + \mathrm{d}u_{g,x})^2} \tag{11}$$

In equation (11), $\mathrm{d}l'$ is the length before deformation and $\mathrm{d}l' = \mathrm{d}x$. The strain of the geosynthetic layer ε_g is

$$\varepsilon_g = \frac{\mathrm{d}l - \mathrm{d}l'}{\mathrm{d}l'} = \sqrt{\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 + \left(1 + \frac{\mathrm{d}u_{g,x}}{\mathrm{d}x}\right)^2} - 1 \tag{12}$$

In (12), the relationship of $\mathrm{d}l' = \mathrm{d}x$ is used. Using Equations (10) and (12), the tension force in geosynthetic layer is

$$T = E_g\varepsilon_g = E_g\left[\sqrt{\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 + \left(1 + \frac{H_b}{G_b}\frac{\mathrm{d}\tau_n{}'}{\mathrm{d}x}\right)^2} - 1\right] \tag{13}$$

where E_g = the tension modulus of geosynthetics in kN/m.

Combining Equations (3), (4), (5), (6) and (13) leads to

$$q - q_s = -(T + T_p)\cos^3\theta\frac{\mathrm{d}^2w}{\mathrm{d}x^2} - \sin\theta\frac{\mathrm{d}T}{\mathrm{d}x} - (H_tG_t + H_bG_b)\frac{\mathrm{d}^2w}{\mathrm{d}x^2} \tag{14}$$

$$\frac{\mathrm{d}^2T}{\mathrm{d}x^2} = \sin\theta\cos\theta\frac{\mathrm{d}^2w}{\mathrm{d}x^2}\frac{\mathrm{d}T}{\mathrm{d}x} + \frac{1}{\cos\theta}\left(\frac{G_t}{H_t} + \frac{G_b}{H_b}\right)\left[\sqrt{\left(\frac{T}{E_g}\right)^2 + 1 - \left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2} - 1\right] \tag{15}$$

Equations (14) and (15) are two second-order non-linear ordinary differential equations with two unknowns T and w.

The spring reaction of soft soils is described by a hyperbolic function. The reaction pressure q_s in (14) is expressed as

$$q_s = \frac{w}{1/k_{s0} + w/q_{sf}} \tag{16}$$

In (16) k_{s0} = the initial tangential spring constant and q_{sf} = the ultimate reaction pressure of the soft soils (springs).

3 Finite difference scheme

The following normalisation may be used to make items in (14) and (15) dimensionless:

$$X = \frac{x}{B},\ W = \frac{w}{B},\ H_t^* = \frac{H_t}{B},\ H_b^* = \frac{H_b}{B}$$

$$G_t^* = \frac{G_t H_t}{k_{so} B^2},\quad G_b^* = \frac{G_b H_b}{k_{so} B^2},\quad E_g^* = \frac{E_g}{k_{so} B^2}$$

$$q^* = \frac{q}{k_{so} B},\quad T_p^* = \frac{T_p}{k_{so} B^2},\quad T_p^* = \frac{T}{k_{so} B^2}$$

$$q^* = \frac{q_s}{k_{so} B},\quad q_{sf}^* = \frac{q_{sf}}{k_{so} B} \tag{17}$$

In (17), B is half width of the pressure boundary (Fig. 5). Using the above normalised items, Equation (16) becomes

$$q_s^* = \frac{W}{1 + W/q_{sf}^*} \tag{18}$$

Equations (14) and (15) can be written as

$$q^* = \frac{W}{1 + W/q_{sf}^*} - \sin\theta \frac{\mathrm{d}T^*}{\mathrm{d}X}$$
$$-[(T^* + T_p^*)\cos^3\theta + (G_t^* + G_b^*)]\frac{\mathrm{d}^2 W}{\mathrm{d}X^2} \tag{19}$$

$$\frac{\mathrm{d}^2 T^*}{\mathrm{d}X^2} = \sin\theta\cos\theta \frac{\mathrm{d}^2 W}{\mathrm{d}X^2}\frac{\mathrm{d}T^*}{\mathrm{d}X}$$
$$+ \frac{1}{\cos\theta}\left(\frac{G_t^*}{H_t^{*2}} + \frac{G_b^*}{H_b^{*2}}\right)\left[\sqrt{\left(\frac{T^*}{E_g^*}\right)^2 + 1} - \left(\frac{\mathrm{d}W}{\mathrm{d}X}\right)^2 - 1\right] \tag{20}$$

where $\sin\theta = (\mathrm{d}W/\mathrm{d}X)/\sqrt{(1+(\mathrm{d}W/\mathrm{d}X)^2}$, $\tan\theta = \mathrm{d}W/\mathrm{d}X$ and $\cos\theta(=1/\sqrt{(1+(\mathrm{d}W/\mathrm{d}X)^2}$.

The following central finite difference scheme is used:

$$\frac{\Delta W}{\Delta X} = \frac{W_{i+1} - W_{i-1}}{2\Delta X}$$
$$\frac{\Delta^2 W}{\Delta X^2} = \frac{W_{i-1} - 2W_i + W_{i+1}}{(\Delta X)^2}$$
$$\frac{\Delta T^*}{\Delta X} = \frac{T_{i+1}^* - T_{i-1}^*}{2\Delta X} \tag{21}$$
$$\frac{\Delta^2 T^*}{\Delta X^2} = \frac{T_{i-1}^* - 2T_i^* + T_{i+1}^*}{(\Delta X)^2}$$

Using (21), Equations (14) and (15) can be written as

$$q_i^* = \frac{W_i}{1 + W/q_{sf}^*} - \left(\sin\theta\frac{\mathrm{d}T^*}{\mathrm{d}X}\right)_i$$
$$-[(T^* + T_p^*)\cos^3\theta + (G_t^* + G_b^*)]_i$$
$$\frac{W_{i-1} - 2W_i + W_{i+1}}{(\Delta X)^2} \tag{22}$$

$$\frac{T_{i-1}^* - 2T_i^* + T_{i+1}^*}{(\Delta X)^2}$$
$$= (\sin\theta\cos\theta)_i \frac{W_{i-1} - 2W_i + W_{i+1}}{(\Delta X)^2}\frac{T_{i+1}^* - T_{i-1}^*}{2\Delta X}$$
$$+ \frac{1}{(\cos\theta)_i}\left(\frac{G_t^*}{H_t^{*2}} + \frac{G_b^*}{H_b^{*2}}\right)_i\left[\sqrt{\left(\frac{T_i^*}{E_g^*}\right)^2 + 1} - \left(\frac{\mathrm{d}W}{\mathrm{d}X}\right)_i^2 - 1\right] \tag{23}$$

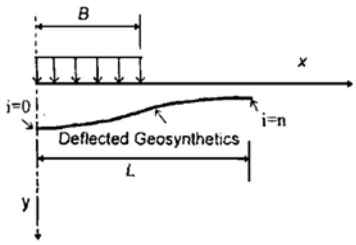

Fig. 5 Half width of symmetric geosynthetic-reinforced foundation

This paper studies the symmetric pressure loading case as shown in Fig. 5. Due to symmetry, only a half width, say right half, is needed in calculation. The right half is divided into n elements, and $\Delta X = (L/B)/n$ with $(n+1)$ node points ($i = 0, 1, \ldots, n$). Due to symmetry, at $X = 0$ (or $x = 0$), the following conditions exist:

$$\mathrm{d}W/\mathrm{d}X = 0 \qquad \mathrm{d}T^*/\mathrm{d}X = 0 \tag{24}$$

At the right end boundary, two boundary conditions can be considered: free or fixed. In this paper, only considered is a free right end at $X = L/B$ (or $x = L$). Since the right end boundary is free, the geosynthetic layer has no tension and the sand layers have no vertical shear forces, so that $T = 0$ and $\tau = \tau' = 0$. According to the Pasternak shear layer assumption, there exist $\tau = G_t \mathrm{d}w/\mathrm{d}x$ and $\tau' = G_b \mathrm{d}w/\mathrm{d}x$. Therefore $\mathrm{d}w/\mathrm{d}x = 0$. Thus at $X = L/B$:

$$T^* = 0 \qquad \mathrm{d}W/\mathrm{d}X = 0 \tag{25}$$

Equations (22) and (23) are a two non-linear equation system which must be solved by iteration. The iteration scheme is, $0 \leqslant i \leqslant n$,

$$(W_0)^{(k+1)} = \frac{1}{2}\left\{2W_1 + TGX_0\left[q_0^* - \frac{W_0}{1 + W_0/q_{sf}^*}\right]\right\}^{(k)} \tag{26}$$

$$(W_i)^{(k+1)} = \frac{1}{2}\left\{W_{i-1} + W_{i+1}\right.$$
$$\left. + TGX_i\left[(\sin\theta)_i\frac{T_{i+1}^* - T_{i-1}^*}{2\Delta X} + q_i^* - \frac{W_i}{1 + W_i/q_{sf}^*}\right]\right\}^{(k)} \tag{27}$$

where

$$TGX_0 = \frac{(\Delta X)^2}{[(T^* + T_p^*) + (G_t^* + G_b^*)]_0}$$
$$TGX_i = \frac{(\Delta X)^2}{[(T^* + T_p^*)\cos^3\theta + (G_t^* + G_b^*)]_i}$$

and

$$(T_0^*)^{(k+1)} = \frac{1}{2}\left\{2T_1^* - XGH_0\left[\sqrt{\left(\frac{T_0^*}{E_g^*}\right)^2 + 1} - 1\right]\right\}^{(k)} \tag{28}$$

$$(T_i^*)^{(k+1)} = \frac{1}{2}\left\{T_{i+1}^* + T_{i-1}^* - (\sin\theta\cos\theta)_i(W_{i-1} - 2W_i + W_{i+1})\frac{T_{i+1}^* - T_{i-1}^*}{2\Delta X} - XGH_i\left[\sqrt{(\frac{T_i^*}{E_g^*})^2 + 1 - (\frac{W_{i+1} - W_{i-1}}{2\Delta X})_i^2} - 1\right]\right\}^{(k)} \quad (29)$$

where

$$T_{n+1}^* = -T_{n-1}^*$$

$$XGH_0 = (\Delta X)^2(\frac{G_t^*}{H_t^{*2}} + \frac{G_b^*}{H_b^{*2}})_0$$

$$XGH_i = \frac{(\Delta X)^2}{(\cos\theta)_i}(\frac{G_t^*}{H_t^{*2}} + \frac{G_b^*}{H_b^{*2}})_i$$

In the above equations the index k is the index of previously computed value and $k+1$ is the index of the new value to be computed. Equations (26) to (29) can be used for iteration computation. The convergence of computation can be checked by $|[W_i^{(k+1)} - W_i^{(k)}]/W_i^{(k)}| \leqslant 10^{-4}$ and $|[T_i^{(k+1)} - T_i^{(k)}]/T_i^{(k)}| < 10^{-4}(i = 0, 1, \cdots n)$. In case of non-symmetric loading, the finite difference scheme above can also be used simply by modifying the boundary condition.

4 Influence of the non-linear spring parameter q_{sf}^*

The influence of the normalised non-linear spring parameter q_{sf}^* on $W - q_s^*$ relationship in Equation (18) can be observed in Fig. 6. The parameter q_{sf}^* is related to the ultimate bearing pressure of soft soil and its non-linear reaction stiffness. When $q_{sf}^* = 10$, the slope of $W - q_s^*$ curve is close to 1, that is, the behaviour is nearly linear elastic. In the case of $q_{sf}^* = 0.1, 0.5, 1, 10$, Equations (26) to (29) have been used to compute settlements and geosynthetic tension as shown in Fig. 7. The results using a linear elastic spring model are shown in Fig. 7 as well. It is found from the fig. that when q_{sf}^* is small, the computed settlement and tension are large. When q_{sf}^* is 10, the computed settlement and tension are close to results using a linear elastic spring model (Yin 1997).

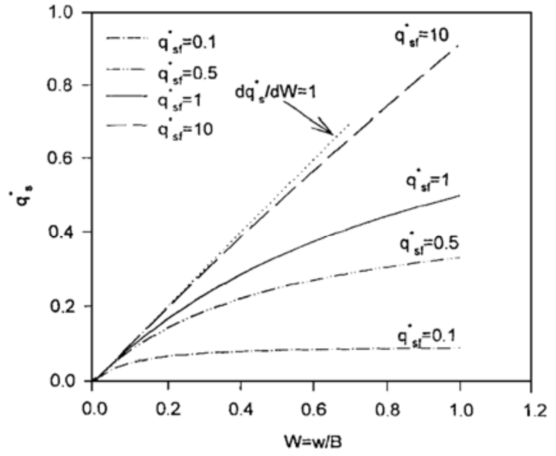

Fig. 6 The influence of parameter q_{sf}^* on $W - q_s^*$ relationship

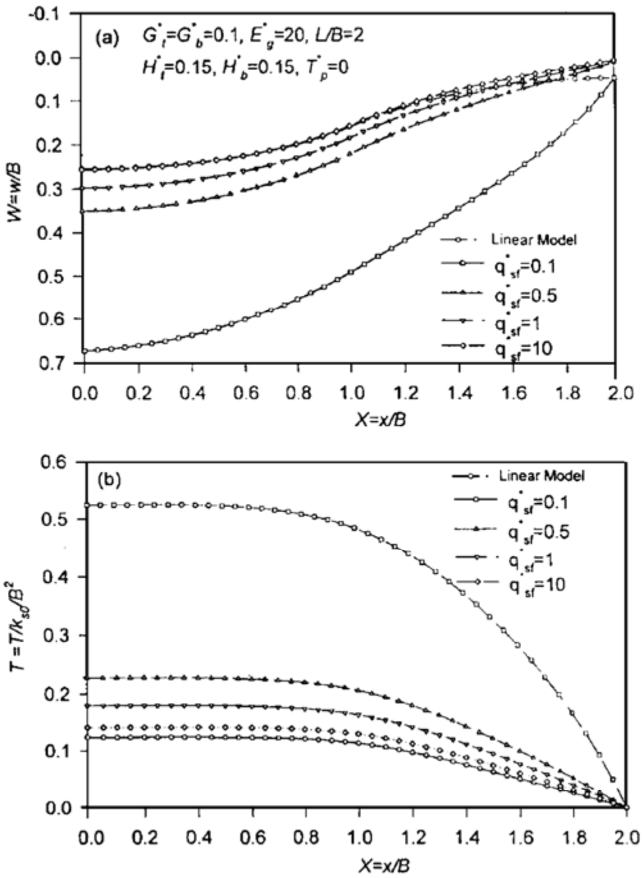

Fig. 7 The influence of parameter q_{sf}^* on settlement and geosynthetic tension

5 Conclusion

This paper derives two governing equations of a non-linear geosynthetic-reinforced foundation model with hyperbolic spring support and presents a finite difference iteration scheme for solutions. The influence of the non-linear spring parameter q_{sf}^* on settlement and tension is studied. It is found that when q_{sf}^* becomes smaller, the settlement and tension get larger. When $q_{sf}^* = 10$ (or larger), the non-linear spring effects can be ignored and computed results are close to the results from a linear spring foundation model.

The proposed 1-D model is based on the Pasternak shear layer assumption. The real deformation and sand/geosynthetic interaction may be more complicated. The proposed model and framework may be extended or modified to consider (a) the elasto-plastic behaviour of sand, (b) the elasto-plastic interaction between sand and geosynthetic layers and (c) multiple layers of geosynthetics.

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