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阶梯方形基础板挠度和内力场的求解

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摘要: 四边自由阶梯方形基础板挠度和内力场的求解极为复杂, 它不仅涉及基础板四自由边界条件的难愈满足, 更涉及阶梯过渡处位移连续性和光滑性的难愈处理。到目前为止, 人们还未给出四边自由阶梯方形基础板在中心垂直荷载作用下挠度和内力场的表达式。本文将载荷与弯曲刚度相除, 提出了载刚比的概念, 巧妙地解决了基础板阶梯过渡处连续性和光滑性难愈处理的问题, 然后将基础板的载刚比展开成双重余弦傅里叶级数, 将挠度函数展开成带补充项的双重余弦傅里叶级数, 在满足板控制方程和四自由边界的条件下, 确定双重余弦傅里叶级数中的待定系数, 获得阶梯方形基础板挠度和内力场的级数表达式, 所得表达式得到有限元数值仿真结果的验证。文中提出的载刚比概念将为非等厚度基础板挠度和内力场的解析求解提供新思路。

关键词: 阶梯方形基础板; 挠度; 内力场; 载刚比的傅里叶级数展开; 阶梯过渡处的处理

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Expressions for deflection and internal forces on ladder square foundation plate

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Abstract: It is very complicated to give the expressions for deflection and internal forces on ladder square foundation plates with four free boundaries because of difficulties in satisfying four free boundaries, displacement continuity and smoothness at ladder transitions. Until now, the expressions of deflection and internal forces on ladder square foundation plates under vertical loads are not available. Herein, by dividing the load with bending stiffness, the concept of load stiffness ratio is put forward, and the problems of continuity and smoothness at ladder transitions are cleverly solved. By the Fourier expansions of load stiffness ratio and plate deflection, the Fourier coefficients of plate deflection of satisfying the plate differential equation and four free boundaries are determined. The expressions for deflection and internal forces on ladder square foundation plates are derived. The expressions are verified by FEM. The proposed concept of load stiffness ratio may provide a new method for solving the deflection and internal forces on ladder square foundation plates.

Key words: ladder square foundation plate; deflection; internal force; Fourier expansion of load stiffness ratio; treatment of ladder transition

0 引言

四边自由方形基础板的弯曲问题是一个复杂的问题, 因为板的解析表达式不仅要求满足四阶弹性挠曲面偏微分方程, 还需要满足四边自由(即四边的弯矩和剪力为零)的力边界条件^[1-2]。对于弹性地基上四边自由的矩形板弯曲问题的求解, 国内外均做了一些研究工作^[3-8], 研究方法主要包括数值法^[9]、解析法^[10]和半数值半解析法^[11-12], 解析法尽管复杂但具有理论价值, 许多学者一直致力于获得四边自由基础板的挠度和内力表达式^[13-14]。卜小明等^[15]直接从 Reissner 板弯曲的基本方程组出发, 以带补充项的傅里叶级数作为挠度和剪力函数, 求出了温克尔地基上自由边矩形

板在受偏心力时弯曲的级数解。叶开沉等采用修正迭代法求解弹性地基正交各向异性阶梯型圆薄板的大挠度问题^[16], 王燮山应用奇异函数及其傅里叶奇延拓展开获得阶梯式变厚度两对边为简支的矩形弹性薄板弯曲问题的通解^[17]。工程中对于阶梯型基础板的力学分析大多数采用有限元法进行数值分析, 到目前为止, 直接利用解析方法给出四边自由边界条件阶梯方形基础板挠度和内力的表达式还没有。本文将载荷与弯曲

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刚度相除, 提出了载刚比的概念, 巧妙地解决了基础板阶梯过渡处的连续性和光滑性问题, 将基础板的载刚比展开成双重余弦傅里叶级数, 将挠度函数展开成带补充项的双重余弦傅里叶级数, 在满足板控制方程和四自由边界的条件下, 确定双重余弦傅里叶级数中的待定系数, 获得阶梯方形基础板挠度和内力场的级数解, 所得表达式得到 ANSYS 数值计算的验证。

1 知识准备

薄板弯曲的基本控制方程为^[18]

$$D(x, y) \nabla^2 \nabla^2 w(x, y) = q(x, y), \quad (1)$$

式中, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, w 为板的挠度, $D = \frac{Et^3}{12(1-\nu^2)}$

为板弯曲刚度, q 为板承受的外载荷集度, t 为板的厚度, E 为板材料的弹性模量, ν 为板材料的泊松比。板截面上承受的内力主要为剪力 (Q_x , Q_y)、弯矩 (M_x , M_y) 和扭矩 (M_{xy}), 如图 1 所示。

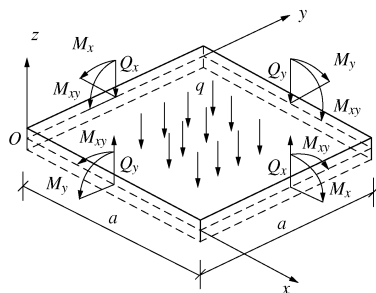


图 1 内力示意图

Fig. 1 Description of internal forces

板的弯矩可通过板的挠度表示为^[18]

$$\left. \begin{aligned} M_x &= -D \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right), \\ M_y &= -D \left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right), \\ M_{xy} &= -D(1-\nu) \frac{\partial^2 w}{\partial x \partial y}. \end{aligned} \right\} \quad (2)$$

板的合成横向剪力与挠度的关系为

$$\left. \begin{aligned} V_x &= -D \left(\frac{\partial^3 w}{\partial x^3} + (2-\nu) \frac{\partial^3 w}{\partial x \partial y^2} \right), \\ V_y &= -D \left(\frac{\partial^3 w}{\partial y^3} + (2-\nu) \frac{\partial^3 w}{\partial x^2 \partial y} \right). \end{aligned} \right\} \quad (3)$$

两直角边交点处的集中反力 R 与挠度的关系为

$$R = 2M_{xy} = -2D(1-\nu) \frac{\partial^2 w}{\partial x \partial y}. \quad (4)$$

对于图 1 所示边界为自由边界, 边上的弯矩和合成横向剪力应为零; 利用式 (2)、(3), 自由边界可以表示成

$$\left. \begin{aligned} \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \Big|_{x=0} &= 0, \\ \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \Big|_{x=a} &= 0, \\ \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \Big|_{y=0} &= 0, \\ \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \Big|_{y=a} &= 0, \end{aligned} \right\} \quad (5)$$

和

$$\left. \begin{aligned} \frac{\partial^3 w}{\partial x^3} + (2-\nu) \frac{\partial^3 w}{\partial x \partial y^2} \Big|_{x=0} &= 0, \\ \frac{\partial^3 w}{\partial x^3} + (2-\nu) \frac{\partial^3 w}{\partial x \partial y^2} \Big|_{x=a} &= 0, \\ \frac{\partial^3 w}{\partial y^3} + (2-\nu) \frac{\partial^3 w}{\partial y \partial x^2} \Big|_{y=0} &= 0, \\ \frac{\partial^3 w}{\partial y^3} + (2-\nu) \frac{\partial^3 w}{\partial y \partial x^2} \Big|_{y=a} &= 0. \end{aligned} \right\} \quad (6)$$

由于四角点处反力为零, 利用式 (4), 得

$$\left. \begin{aligned} \frac{\partial^2 w}{\partial x \partial y} \Big|_{x=0, y=0} &= 0, \\ \frac{\partial^2 w}{\partial x \partial y} \Big|_{x=0, y=a} &= 0, \\ \frac{\partial^2 w}{\partial x \partial y} \Big|_{x=a, y=0} &= 0, \\ \frac{\partial^2 w}{\partial x \partial y} \Big|_{x=a, y=a} &= 0. \end{aligned} \right\} \quad (7)$$

满足控制方程 (1), 并满足四边自由的边界条件 (5), (6)、(7) 的平板弯曲是一个复杂的数学问题, 如果成为图 2 所示的阶梯方形基础板问题, 则不仅涉及四自由边界条件的处理, 还涉及阶梯四过渡处位移连续性和光滑性的处理。尽管阶梯型基础板弯曲问题的研究具有理论价值, 但由于问题求解的复杂性, 到目前为止, 人们还没有给出四边自由阶梯方形基础板内力场和位移场的解析表达式或级数表达式。本文提出了载刚比的概念, 巧妙地解决了基础板阶梯过渡处的连续性和光滑性问题, 在满足阶梯方形基础板控制方程和四自由边界的条件下, 获得阶梯方形基础板挠度和内力场的级数表达式, 所得表达式得到 ANSYS 数值计算结果的验证。

2 阶梯方形基础板载刚比的傅里叶表示

取一阶梯正方形基础板如图 2 所示, 板中间较厚的部位和旁边较薄的部位厚度分别为 t_2 和 t_1 , 其宽度

分别为 b 和 a , 板中心作用垂直集中荷载 F , 板下底面承受的均匀分布荷载为 $q = F/a^2$ 。基本控制方程 (1) 可以改写成

$$\nabla^2 \nabla^2 w(x, y) = p(x, y) \quad , \quad (8)$$

定义

$$p(x, y) = \frac{q(x, y)}{D(x, y)} \quad (9)$$

为基础板的载刚比, 这里

$$D(x, y) = \begin{cases} \frac{Et_1^3}{12(1-\nu^2)} \stackrel{\text{def}}{=} D_1 & ((x, y) \in A_1 \setminus A_2) \\ \frac{Et_2^3}{12(1-\nu^2)} \stackrel{\text{def}}{=} D_2 & ((x, y) \in A_2) \end{cases} \quad (10)$$

$$\left. \begin{aligned} A_1 &= \{(x, y) | 0 < x < a/2, 0 < y < a/2\} , \\ A_2 &= \{(x, y) | h < x < h+b/2, h < y < h+b/2\} , \\ h &= (a-b)/2 . \end{aligned} \right\} \quad (11)$$

考虑板基础和载荷的对称性, 这里仅考虑基础板的 1/4 区域。

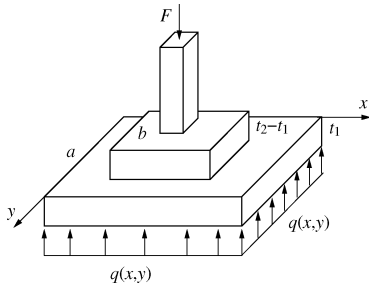


图2 阶梯方形基础板的尺寸和载荷

Fig. 2 Sizes and loads of ladder square plates

阶梯型方形基础板取隔离体如图3所示, 在 $x = h$ 截面上的单位长度平均剪力

$$V_x = \frac{1}{4b} \frac{F}{a^2} (a^2 - b^2) \quad , \quad (12)$$

而总弯矩为

$$M_x = \frac{1}{2} \frac{F}{a^2} ah^2 \quad . \quad (13)$$

用 M_1 表示作用在 $x = h$ 截面上非台阶过度处的总弯矩, 用 M_2 表示台阶过度处的总弯矩, 它们满足

$$M_x = M_1 + M_2 = \frac{Fh^2}{2a} \quad (14)$$

和

$$\left. \begin{aligned} M_1 &= \frac{EI_1}{\rho} , \\ M_2 &= \frac{EI_2}{\rho} . \end{aligned} \right\} \quad (15)$$

在 $x = h$ 台阶过度处和非过度处应该具有同样的弯曲曲率 $1/\rho$, 其中,

$$\left. \begin{aligned} I_1 &= \frac{1}{12} (a-b)t_1^3 , \\ I_2 &= \frac{1}{12} bt_2^3 , \end{aligned} \right\} \quad (16)$$

分别为截面 $x = h + 0^+$ 上台阶非阶梯过度处和阶梯过度处截面惯性矩。联立 (14) ~ (16), 有

$$M_2 = \frac{Fh^2}{2a \left(\frac{a}{b} - 1 \right) \frac{t_1^3}{t_2^3} + 2a} \quad . \quad (17)$$

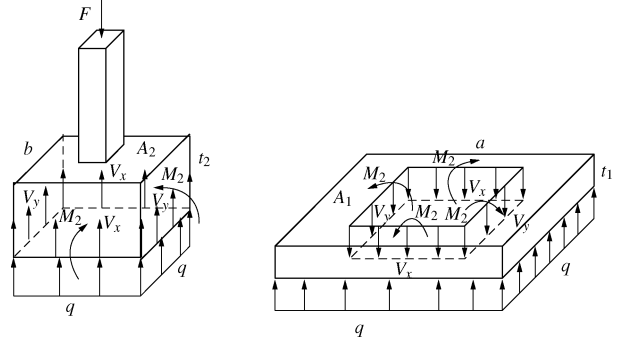


图3 阶梯方形基础板隔离体

Fig. 3 Isolation bodies of ladder square plates

式 (9) 中基础板 1/4 区域的载刚比可表示成

$$p(x, y) = \frac{q_1(x, y)}{D_1} \chi_{A_1 \setminus A_2}(x, y) + \frac{q_2(x, y)}{D_2} \chi_{A_2}(x, y) \quad , \quad (18)$$

A_1 和 A_2 在式 (11) 给出, 特征函数 $\chi_A(x, y)$ 定义为

$$\chi_A(x, y) = \begin{cases} 1 & ((x, y) \in A) \\ 0 & ((x, y) \notin A) \end{cases} \quad , \quad (19)$$

考虑到阶梯基础板的厚度不一样, 为了利用 (18), 在阶梯过度处的相互作用内力 (弯矩和剪力) 不得不通过截面法转变成外力显示出来 (它们分别作用在厚板的截面上和薄板的截面上; 参考图3); 因此, 基础板 1/4 区域的荷载集度表示成

$$\begin{aligned} q_1(x, y) = & -\frac{F}{a^2} + \frac{F}{4b} \left(1 - \frac{b^2}{a^2} \right) \left[\delta(x-h) \chi_{h \leq y \leq \frac{a}{2}} + \right. \\ & \left. \delta(y-h) \chi_{h \leq x \leq \frac{a}{2}} \right] + \frac{M_2}{db} \left[\delta(x-(h-d)) \chi_{h \leq y \leq \frac{a}{2}} - \right. \\ & \left. \delta(x-h) \chi_{h \leq y \leq \frac{a}{2}} \right] + \frac{M_2}{db} \left[\delta(y-(h-d)) \chi_{h \leq x \leq \frac{a}{2}} - \right. \\ & \left. -\delta(y-h) \chi_{h \leq x \leq \frac{a}{2}} \right] \quad ((x, y) \in A_1 \setminus A_2) \quad . \quad (20) \end{aligned}$$

$$\begin{aligned} q_2(x, y) = & -\frac{F}{a^2} + \delta \left(x - \frac{a}{2} \right) \delta \left(y - \frac{a}{2} \right) \frac{F}{4} - \\ & \frac{F}{4b} \left(1 - \frac{b^2}{a^2} \right) \left[\delta(x-h) \chi_{h \leq y \leq \frac{a}{2}} + \delta(y-h) \chi_{h \leq x \leq \frac{a}{2}} \right] + \\ & \frac{M_2}{db} \left[\delta(x-(h+d)) \chi_{h \leq y \leq \frac{a}{2}} - \delta(x-h) \chi_{h \leq y \leq \frac{a}{2}} \right] + \\ & \frac{M_2}{db} \left[\delta(y-(h+d)) \chi_{h \leq x \leq \frac{a}{2}} - \delta(y-h) \chi_{h \leq x \leq \frac{a}{2}} \right] \\ & ((x, y) \in A_2) \quad . \quad (21) \end{aligned}$$

这里假设单位长度上的剪力 V_x 和弯矩 M_2/b 在

$h \leq y \leq a/2$ 间均匀分布, 弯矩 M_2 由一对大小相等、方向相反、距离为 d 的荷载 M_2/d 代替, $\delta(x-h)$ 函数满足

$$\delta(x-h)\chi_{h \leq y \leq \frac{a}{2}} = \begin{cases} \infty & \left(\text{当 } x=h \text{ 和 } h \leq y \leq \frac{a}{2} \right), \\ 0 & \text{(其他)} \end{cases} \quad (22)$$

$$\int_{-\infty}^{\infty} f(x)\delta(x-h)dx = f(h) \quad (23)$$

式(18)中的载刚比可展开成双重余弦傅里叶级数

$$p(x,y) = \sum_{m=0,2,4,\dots} \sum_{n=0,2,4,\dots} p_{mn} \cos \beta_m x \cos \beta_n y, \quad (24)$$

式中, $\beta_m = \frac{m\pi}{a}$, $\beta_n = \frac{n\pi}{a}$, 展开系数 p_{mn} 与式(18)

载刚比 $p(x,y)$ 的关系为

$$\begin{aligned} p_{mn} &= \frac{16\varepsilon_m \varepsilon_n}{a^2} \int_{A_1 \setminus A_2} \frac{q_1}{D_1} \cos \beta_m x \cos \beta_n y dx dy + \\ &\quad \frac{16\varepsilon_m \varepsilon_n}{a^2} \int_{A_2} \frac{q_2}{D_2} \cos \beta_m x \cos \beta_n y dx dy \\ &= \frac{16\varepsilon_m \varepsilon_n}{D_1 a^2} \left(I_{mn} + II_{mn} + \frac{D_1}{D_2} III_{mn} \right) \quad (25) \end{aligned}$$

基础板结构和载荷的对称性使得积分仅需在 1/4 区域进行, I_{mn} 、 II_{mn} 、 III_{mn} 为图 4 所示 1/4 区域再划分成 3 区域的分别积分结果, 其中

$$\varepsilon_n = \begin{cases} 1/2 & (n=0) \\ 1 & (n=2,4,6,\dots) \end{cases}, \quad (26)$$

$$\left. \begin{aligned} I_{mn} &= I_{mn}^{(1)} + I_{mn}^{(2)} \\ II_{mn} &= II_{mn}^{(1)} + II_{mn}^{(2)} \\ III_{mn} &= III_{mn}^{(1)} + III_{mn}^{(2)} \end{aligned} \right\} \quad (27)$$

其结果为

$$\begin{aligned} I_{mn}^{(1)} &= \int_0^{a/2} \int_0^h \left\{ -\frac{F}{a^2} + \frac{F}{4b} \left(1 - \frac{b^2}{a^2} \right) \delta(y-h) \chi_{h \leq x \leq \frac{a}{2}} \right\} \\ &\quad \cos \beta_m x \cos \beta_n y dx dy = -\frac{F}{a^2 \beta_m \beta_n} \sin \left(\beta_m \frac{a}{2} \right) \sin(\beta_n h) + \\ &\quad \frac{F}{4b \beta_m} \left(1 - \frac{b^2}{a^2} \right) \left[\sin \left(\beta_m \frac{a}{2} \right) - \sin(\beta_m h) \right] \cos(\beta_n h), \quad (28) \end{aligned}$$

$$\begin{aligned} I_{mn}^{(2)} &= \int_0^{a/2} \int_0^h \frac{M_2}{db} \left[\delta(y-h+d) \chi_{h \leq x \leq \frac{a}{2}} - \right. \\ &\quad \left. \delta(y-h) \chi_{h \leq x \leq \frac{a}{2}} \right] \cos \beta_m x \cos \beta_n y dx dy \\ &= \frac{M_2 \beta_n \sin(\beta_n h)}{\beta_m b} \left(\sin \frac{\beta_m a}{2} - \sin \beta_m h \right), \quad (29) \end{aligned}$$

$$\begin{aligned} II_{mn}^{(1)} &= \int_0^h \int_h^{a/2} \left[-\frac{F}{a^2} + \frac{F}{4b} \left(1 - \frac{b^2}{a^2} \right) \cdot \delta(x-h) \chi_{h \leq y \leq \frac{a}{2}} \right] \\ &\quad \cos \beta_m x \cos \beta_n y dy dx = -\frac{F}{a^2 \beta_m \beta_n} \sin(\beta_m h) \end{aligned}$$

$$\begin{aligned} &\left[\sin(\beta_n a/2) - \sin(\beta_n h) \right] + \frac{F}{4b \beta_n} \left(1 - \frac{b^2}{a^2} \right) \cdot \\ &\quad [\sin(\beta_n a/2) - \sin(\beta_n h)] \cos \beta_m h, \quad (30) \end{aligned}$$

$$\begin{aligned} II_{mn}^{(2)} &= \int_0^h \int_h^{a/2} \frac{M_2}{db} \left[\delta(x-h+d) \chi_{h \leq y \leq \frac{a}{2}} - \right. \\ &\quad \left. \delta(x-h) \chi_{h \leq y \leq \frac{a}{2}} \right] \cos \beta_m x \cos \beta_n y dy dx \\ &= \frac{M_2 \beta_m}{b \beta_n} \sin \beta_m h \left[\sin \frac{\beta_n a}{2} - \sin(\beta_n h) \right], \quad (31) \end{aligned}$$

$$\begin{aligned} III_{mn}^{(1)} &= \int_h^{a/2} \int_h^{a/2} \left\{ -\frac{F}{a^2} + \delta \left(x - \frac{a}{2} \right) \delta \left(y - \frac{a}{2} \right) \frac{F}{4} - \right. \\ &\quad \left. \frac{F}{4b} \left(1 - \frac{b^2}{a^2} \right) \left[\delta(y-h) \chi_{h \leq x \leq \frac{a}{2}} + \delta(x-h) \cdot \right. \right. \\ &\quad \left. \left. \chi_{h \leq y \leq \frac{a}{2}} \right] \right\} \cos \beta_m x \cos \beta_n y dy dx \\ &= -\frac{F}{a^2 \beta_m \beta_n} \left(\sin \frac{\beta_m a}{2} - \sin \beta_m h \right) \left(\sin \frac{\beta_n a}{2} - \sin \beta_n h \right) + \\ &\quad \frac{F}{4} \cos \frac{m\pi}{2} \cos \frac{n\pi}{2} - \frac{F}{4b} \left(1 - \frac{b^2}{a^2} \right) \cdot \\ &\quad \left[\frac{\left(\sin \frac{\beta_m a}{2} - \sin \beta_m h \right) \cos \beta_n h}{\beta_m} + \frac{\cos \beta_m h \left(\sin \frac{\beta_n a}{2} - \sin \beta_n h \right)}{\beta_n} \right], \quad (32) \end{aligned}$$

$$\begin{aligned} III_{mn}^{(2)} &= \int_h^{a/2} \int_h^{a/2} \left\{ \frac{M_2}{db} \left[\delta(x-h-d) \chi_{h \leq y \leq \frac{a}{2}} - \right. \right. \\ &\quad \left. \delta(x-h) \chi_{h \leq y \leq \frac{a}{2}} \right] + \frac{M_2}{db} \left[\delta(y-h-d) \chi_{h \leq x \leq \frac{a}{2}} - \right. \\ &\quad \left. \delta(y-h) \chi_{h \leq x \leq \frac{a}{2}} \right] \right\} \cos \beta_m x \cos \beta_n y dy dx \\ &= \frac{M_2}{b} \left[\frac{\beta_m \sin \beta_m h}{\beta_n} \left(-\sin \frac{\beta_n a}{2} + \sin \beta_n h \right) + \right. \\ &\quad \left. \frac{\beta_n \sin \beta_n h}{\beta_m} \left(-\sin \frac{\beta_m a}{2} + \sin \beta_m h \right) \right] \quad (33) \end{aligned}$$

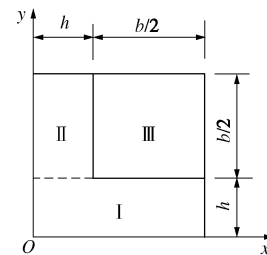


图 4 阶梯方形基础板 1/4 区域的积分区域划分

Fig. 4 Integral region division of ladder square plates

本节提出了载刚比的概念, 把阶梯过渡处的相互

作用弯矩 M_2 由一对大小相等、方向相反、距离为 d , 垂直板面的荷载 M_2/d 代替, 并将载刚比展开成傅里叶级数。在式 (29)、(31)、(33) 的简化过程中, 令 $d \rightarrow +0$, 利用导数的定义巧妙地消除了引入的两垂直荷载 M_2/d 间的距离量 d 。

3 阶梯方形基础板挠度和内力的级数表达式

在对称荷载作用下, 阶梯方形基础板的挠度关于 $x=a/2$ 和 $y=a/2$ 是对称的, 即

$$w(x, y) = w(a-x, y) = w(x, a-y), \quad (34)$$

为了在边界条件 (5) ~ (7) 下求解式 (8), 将挠度设为带有补充项的双重余弦傅里叶展开级数^[15]:

$$w(x, y) = \sum_{m=0,2,4,\dots}^{\infty} \sum_{n=0,2,4,\dots}^{\infty} w_{mn} \cos \beta_m x \cos \beta_n y + \sum_{m=0,2,4,\dots}^{\infty} [(2-\nu)\beta_m^2 f_1(y) + f_2(y)] C_m \cos \beta_m x + \sum_{n=0,2,4,\dots}^{\infty} [(2-\nu)\beta_n^2 f_1(x) + f_2(x)] G_n \cos \beta_n y, \quad (35)$$

式中,

$$\left. \begin{aligned} f_1(y) &= \frac{4ay^3 - 2a^2y^2 - 2y^4}{24a^2}, \\ f_2(y) &= \frac{2ay - 2y^2}{2a^2}. \end{aligned} \right\} \quad (36)$$

考虑到存在

$$\cos \beta_m(a-x) = \cos(m\pi - \beta_m x) = \cos(\beta_m x), m \in \text{even}, \quad \left. \begin{aligned} f_1(y) &= f_1(a-y), \\ f_2(y) &= f_2(a-y), \end{aligned} \right\} \quad (37)$$

对称性条件 (34) 已经满足。

阶梯方形基础板的荷载和结构相对于方形对角线对称, 满足

$$w(x, y) = w(y, x), \quad (38)$$

它导致 (35) 式中的系数具有以下关系

$$\left. \begin{aligned} w_{mn} &= w_{nm}, \\ C_m &= G_m. \end{aligned} \right\} \quad (39)$$

因此, 挠度场 (35) 可以改写成

$$w(x, y) = \sum_{m=0,2,4,\dots}^{\infty} \sum_{n=0,2,4,\dots}^{\infty} w_{mn} \cos \beta_m x \cos \beta_n y + \sum_{m=0,2,4,\dots}^{\infty} [(2-\nu)\beta_m^2 f_1(y) + f_2(y)] C_m \cos \beta_m x + \sum_{n=0,2,4,\dots}^{\infty} [(2-\nu)\beta_n^2 f_1(x) + f_2(x)] C_n \cos \beta_n y, \quad (40)$$

式中, w_{mn} , C_m 为待定系数。

容易验证, 不管待定系数 w_{mn} , C_m 为何值, 式 (40) 中的 $w(x, y)$ 满足边界条件 (6) 和 (7)。现 $w(x, y)$ 只需要满足边界条件 (5) 和基础板控制方程 (8)。满足

边界条件 (5) 给出以下关系:

$$\begin{aligned} \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \Big|_{x=0} = & \sum_{n=0,2,4,\dots}^{\infty} \left[\sum_{m=0,2,4,\dots}^{\infty} -w_{mn} (\beta_m^2 + \nu \beta_n^2) \right] \cos \beta_n y + \\ & \sum_{m=0,2,4,\dots}^{\infty} -(2-\nu) [\beta_m^4 f_1(y) - \nu \beta_m^2 f_1''(y)] C_m + \\ & \sum_{m=0,2,4,\dots}^{\infty} [-\beta_m^2 f_2(y) + \nu f_2''(y)] C_m + \\ & \sum_{n=0,2,4,\dots}^{\infty} [(2-\nu) \beta_n^2 (f_1''(0) - \nu \beta_n^2 f_1(0)) + \\ & f_2''(0) - \nu \beta_n^2 f_2(0)] C_n \cos \beta_n y = 0. \end{aligned} \quad (41)$$

对式 (41) 中的 $f_1(y)$ 和 $f_2(y)$ 及其导数实施傅里叶级数展开:

$$\left. \begin{aligned} f_1(y) &= \sum_{n=0,2,4,\dots}^{\infty} a_n^{(0)} \cos \beta_n y, a_0^{(0)} = \frac{-a^2}{360}, a_n^{(0)} = \frac{4}{a^2 \beta_n^4}, \\ f_2(y) &= \sum_{n=0,2,4,\dots}^{\infty} b_n^{(0)} \cos \beta_n y, b_0^{(0)} = \frac{1}{6}, b_n^{(0)} = \frac{-4}{a^2 \beta_n^2}, \\ f_1''(y) &= \sum_{n=0,2,4,\dots}^{\infty} a_n^{(2)} \cos \beta_n y, a_0^{(2)} = 0, a_n^{(2)} = \frac{-4}{a^2 \beta_n^2}, \\ f_2''(y) &= \sum_{n=0,2,4,\dots}^{\infty} b_n^{(2)} \cos \beta_n y, b_n^{(2)} = -\frac{2}{a^2} \delta_{0n}, \\ f_1^{(4)}(y) &= \sum_{n=0,2,4,\dots}^{\infty} a_n^{(4)} \cos \beta_n y, a_n^{(4)} = \frac{-2}{a^2} \delta_{0n}, \\ f_2^{(4)}(y) &= 0. \end{aligned} \right\} \quad (42)$$

结合式 (41)、(42) 可建立以下关于 w_{mn} , C_m 线性方程组

$$\begin{aligned} \sum_{m=0,2,4,\dots}^{\infty} -w_{mn} (\beta_m^2 + \mu \beta_n^2) + \sum_{m=0,2,4,\dots}^{\infty} -(2-\nu) [\beta_m^4 a_n^{(0)} - \\ \nu \beta_m^2 a_n^{(2)}] C_m + \sum_{m=0,2,4,\dots}^{\infty} [-\beta_m^2 b_n^{(0)} + \nu b_n^{(2)}] C_m + [(2-\nu) \times \\ \beta_n^2 (f_1''(0) - \nu \beta_n^2 f_1(0)) + f_2''(0) - \nu \beta_n^2 f_2(0)] C_n = 0 \\ (n=0,2,4,\dots). \end{aligned} \quad (43)$$

将式 (40) 代入式 (8) 的左边, 得

$$\begin{aligned} \nabla^2 \nabla^2 w = & \sum_{m=0,2,4,\dots}^{\infty} \sum_{n=0,2,4,\dots}^{\infty} w_{mn} (\beta_m^2 + \beta_n^2)^2 \cos \beta_m x \cos \beta_n y + \\ & \sum_{m=0,2,4,\dots}^{\infty} \sum_{n=0,2,4,\dots}^{\infty} \{ (2-\nu) [\beta_m^6 a_n^{(0)} + \beta_m^2 a_n^{(4)} - 2\beta_m^4 a_n^{(2)}] + \\ & \beta_m^4 b_n^{(0)} - 2\beta_m^2 b_n^{(2)} \} C_m \cos \beta_m x \cos \beta_n y + \\ & \sum_{m=0,2,4,\dots}^{\infty} \sum_{n=0,2,4,\dots}^{\infty} \{ (2-\nu) [\beta_n^6 a_m^{(0)} + \beta_n^2 a_m^{(4)} - 2\beta_n^4 a_m^{(2)}] + \\ & \beta_n^4 b_m^{(0)} - 2\beta_n^2 b_m^{(2)} \} C_n \cos \beta_m x \cos \beta_n y. \end{aligned} \quad (44)$$

将式 (44) 和 (24) 代入式 (8), 考虑等式两边有相同的展开基且对任意 x 和 y 成立, 形成关于 w_{mn} , C_m 线性方程组

$$w_{mn}(\beta_m^2 + \beta_n^2)^2 + \{(2-\nu)[\beta_m^6 a_n^{(0)} + \beta_m^2 a_n^{(4)} - 2\beta_m^4 a_n^{(2)}] + \beta_m^4 b_n^{(0)} - 2\beta_m^2 b_n^{(2)}\}C_m + \{(2-\nu)[\beta_n^6 a_m^{(0)} + \beta_n^2 a_m^{(4)} - 2\beta_n^4 a_m^{(2)}] + \beta_n^4 b_m^{(0)} - 2\beta_n^2 b_m^{(2)}\}C_n = p_{mn}, \quad m=0,2,4,\dots; n=0,2,4,\dots \quad (45)$$

联立线性方程组(43)和(45)可以解出 w_{mn} 和 C_m , 将这些 w_{mn} 和 C_m 代入式(40)最终获得阶梯方形基础板的挠度场 $w(x,y)$, 进一步将该 $w(x,y)$ 代入式(2)和(3)可以求得方形基础底板各点处的弯矩 (M_x, M_y)、扭矩 (M_{xy}) 和合成剪力 (V_x, V_y)。

4 阶梯方形基础板挠度和内力场级数解的有限元数值验证

图2所示阶梯方形基础板的外围边长 a 、阶梯过渡处边长 b 、旁边处厚度 t_1 、中间处厚度 t_2 、弹性模量 E 、泊松比 ν 、作用荷载 F 的取值为 $a=4\text{ m}$, $b=2\text{ m}$, $t_1=0.1\text{ m}$, $t_2=0.2\text{ m}$, $E=26\text{ GPa}$, $\nu=0.167$, $F=100\text{ kN}$, 基础板底部接触力集度为

$$q(x,y) = F/a^2. \quad (46)$$

把各参数值代入式(43)和(45), 解关于 w_{mn} 和 C_m 的线性方程, 将解代入(40), 得该阶梯方形基础板的挠度场 $w(x,y)$, 其结果如图5(a)所示; 进一步将挠度场 $w(x,y)$ 代入(2), 得阶梯方形基础板的弯矩场 M_x , 其结果如图5(b)~(e)所示。同时, 通过 ANSYS 的 solid45 实体单元对该阶梯方形基础板进行有限元数值计算, 在板中央加 100 kN 的集中荷载, 板下表面施加大小为 6250 Pa 的均布荷载, 均布荷载与集中荷载在 z 方向平衡。为了便于比较, 这里将有限元数值仿真结果都示于图5(a)~5(e)中, 级数表达式的结果与有限元的仿真结果相吻合。

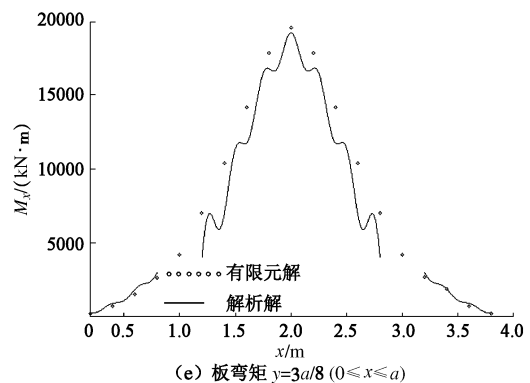
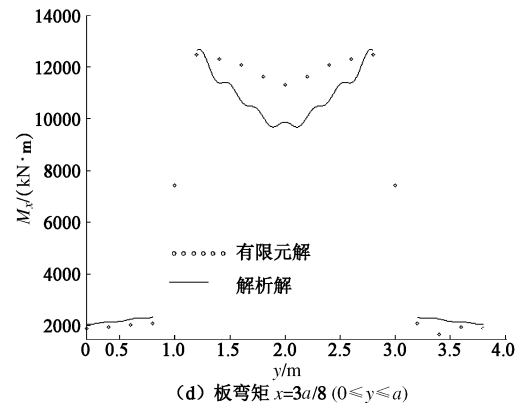
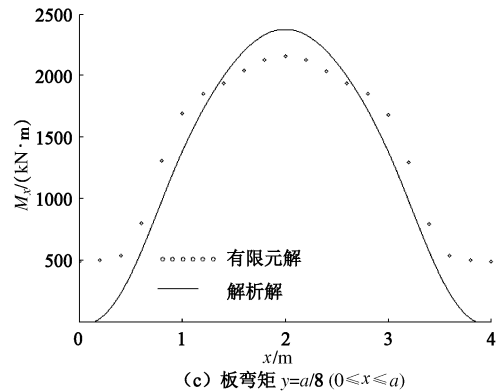
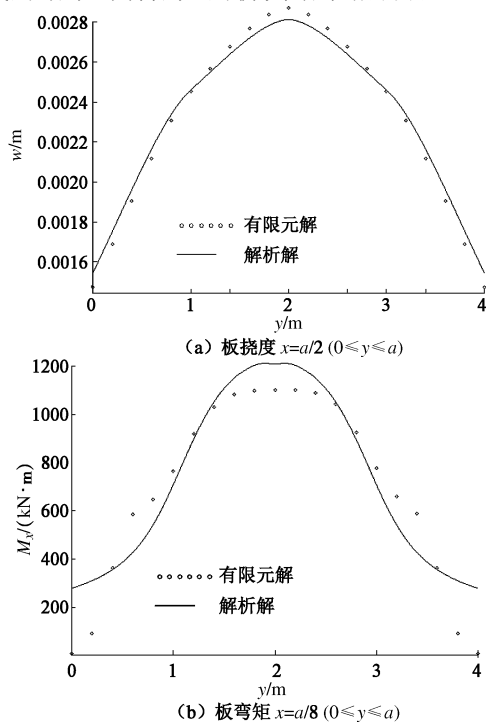


图5 解析解与有限元数值结果的比较

Fig. 5 Comparison between analytical solutions and FEM results

5 结 语

四边自由阶梯方形基础板的挠度和内力场求解极为复杂, 人们目前还未给出四边自由阶梯方形基础板在中心垂直荷载作用下挠度和内力场的级数表达式。通过提出基础板载刚比的概念, 本文巧妙地解决了基础板阶梯过渡处的连续性和光滑性难愈处理的问题, 获得具有理论价值的阶梯方形基础板挠度和内力场的级数表达式, 所得表达式得到有限元数值仿真结果的验证, 这里提出的载刚比概念将为非等厚度基础板挠度和内力场的求解提供新思路。

考虑基础板的结构对称性, 偏心荷载可以向板中心简化, 分解成为对称荷载(作用于基础板中心的垂直荷载)和反对称荷载(作用于基础板中心的力偶矩), 我们将在今后开展阶梯方形基础板在反对称荷载作用

下的挠度和内力的求解工作。

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