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1-D consolidation of a single soil layer with depth-dependent stress by multi-stage loading

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Abstract: In geotechnical engineering, when the strength of soft soils is relatively low, a rapid loading rate can lead to ground failure. In this situation, a multi-stage loading scheme can be utilized to achieve higher soil strength by consolidating the soil layer to a certain degree before applying the next, larger load(s). Additionally, the total stress in the soil layer usually varies, and, in many cases, this variation is not uniform with depth, for example, when the loading is applied within a small area over a thick soil layer. In this study, a thorough, explicit analytical solution is presented for the consolidation of a single soil layer using a multi-stage loading with depth-dependent total stress. The particular case of a two-stage loading scheme is selected to investigate the consolidation behavior of a soil layer. Finally, the convergence of the analytical solution is assessed by comparing the calculated results using the various terms of the series to facilitate the use of the solution by engineers and to provide sufficient accuracy.

Key words: soil layer; one-dimensional consolidation; excess pore water pressure; multi-stage loading; depth-dependent stress; analytical solution

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多级加载下附加应力沿深度变化的单层地基一维固结

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摘　要：在岩土工程中，当天然土层的强度较小时，加载速率过快可能会导致地基失稳。在这种情况下，可以采用多级加载的方式来逐步提高地基强度，使地基在施加下一级更大的荷载时已固结到一定程度。另外，在很多情况下，外部荷载在地基内引起的总附加应力总是随着深度发生变化，而并非保持不变，例如，当土层中地下水位发生变化时即是如此。详细给出了一个考虑多级加载下附加应力沿深度变化的单层地基一维固结的完全显式解析解。然后，选取外部荷载二级施加的特例来分析单层地基的固结性状。最后，通过比较不同的计算项数所得的计算结果来评估所得的级数解的收敛性，以利于工程人员采用本文解时能够获得足够的精度。

关键词：土层；一维固结；超静孔压；多级加载；非均布应力；解析解

0 Introduction

Since the classical one-dimensional consolidation theory was proposed by Terzaghi[1], a large number of studies have been performed to extend it to account for more realistic field situations by modifying the various assumptions in the original model. Schiffman and Stein et al.[2] developed an analytical solution for the one-dimensional consolidation of a layered system by

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considering a general set of boundary conditions and an arbitrary loading history. However, the involved computation is very tedious. In this context, a more explicit and tractable analytical solution was provided by Lee et al.[3] for predicting the consolidation behavior of a layered system. Olson[4] extended Terzaghi's one-dimensional consolidation theory by incorporating a simple ramp loading in lieu of instantaneous loading. In Olson's solution, the total stress caused by the external loads was assumed to be uniform with depth. However, in many cases, the total stress due to the external load varies with depth. Therefore, by incorporating a depth-dependent ramp loading, Zhu et al.[5-6] obtained a series of analytical solutions for the one-dimensional consolidation of a single soil layer and a double soil layer. Differing from the previous theories, the stress within the soil layer incorporated in their studies depended simultaneously on both time and depth. Xie et al.[7-8] later presented a series of analytical solutions for the consolidation of a double soil layer dealing with the situations of partially drained boundaries and nonlinearity during consolidation. In view of the time-dependent characteristics of the external load, Conte et al.[9-10] presented two analytical solutions for the linear and the nonlinear consolidation of a single soil layer based on a Fourier series, respectively. By choosing a suitable period for the Fourier series, the solution can be adapted to model a variety of loading schemes, such as cyclic loading and single-stage loading. Based on the Biot's governing consolidation equations for transversely isotropic clay, Cai et al.[11] investigated the axisymmetric consolidation of a semi-infinite transversely isotropic saturated clay subjected to a time-varying circular pressure at the ground surface by employing the Laplace－Hankel transform technique. In recent years, many studies have been conducted to deal with the one-dimensional consolidation problem with non-Darcian flow. For example, Xie et al.[12] presented an approximate analytical solution for the consolidation of a soil layer with consideration of the threshold gradient under a time-varying loading. Li et al.[13-14] employed a numerical approach to investigate the consolidation of a single soil layer and a double-layered soil, respectively, by considering a non-Darcian flow law described by exponent and threshold.

In practical engineering, such as in the construction of buildings or embankments on clayey soils, a rapid loading rate may lead to the failure of the clayey soil. In this situation, a multi-stage loading scheme can be utilized to achieve higher soil strength by consolidating the soil layer to a certain degree before applying the next, larger load(s). Additionally, in many cases where the stress in the soil layer varies with depth, this variation is not uniformly distributed with depth. Many factors can lead to such a varying stress, such as the use of a relatively small loading area over a very thick soil layer. Therefore, when the external load is gradually applied stage by stage over time, the stress in the soil layer should be a function simultaneously depending on both time and depth. In this study, a thorough, explicit analytical solution is developed for the consolidation of a single soil layer with a depth-dependent stress by multi-stage loading. The particular case of a two-stage loading scheme is selected to investigate the consolidation behavior of the soil layer, and the convergence of the solution is assessed. The results show that the solution is very simple for engineers to use because of its rapid convergence with sufficient accuracy.

1 Basic equation and solutions

For one-dimensional consolidation of a single soil layer with a pervious top surface and an impervious bottom, as shown in Fig. 1(b), all of the assumptions in Terzaghi[1] are followed in this derivation except that a multi-stage and depth-dependent stress, given by Eqs. (1) and (2), respectively, is included in this study.

$$\sigma(z,t)=\begin{cases}\sigma_{n-1}(z)+\dfrac{t-t_{2n-2}}{t_{2n-1}-t_{2n-2}}[\sigma_n(z)-\sigma_{n-1}(z)] & (t_{2n-2}\leqslant t\leqslant t_{2n-1})\\ \sigma_n(z) & (t_{2n-1}\leqslant t\leqslant t_{2n})\end{cases}, \quad (1)$$

where

$$\sigma_n(z)=\sigma_{n\mathrm{T}}+(\sigma_{n\mathrm{B}}-\sigma_{n\mathrm{T}})\frac{z}{H}, \quad (2)$$

and $\sigma_n(z)$ is the final stress varying with depth caused by the n^{th} stage loading, with values at the top surface and bottom base of $\sigma_{n\mathrm{T}}$ and $\sigma_{n\mathrm{B}}$, respectively; t_{2n-2}, t_{2n} are the final times of the $(n-1)^{\text{th}}$ and n^{th} stage loadings, respectively; t_{2n-1} corresponds to the time when the n^{th} stage loading is increased to the final value; and H is the thickness of the soil layer ($n=1, 2, 3, \cdots$; $\sigma_0=0$; $t_0=0$).

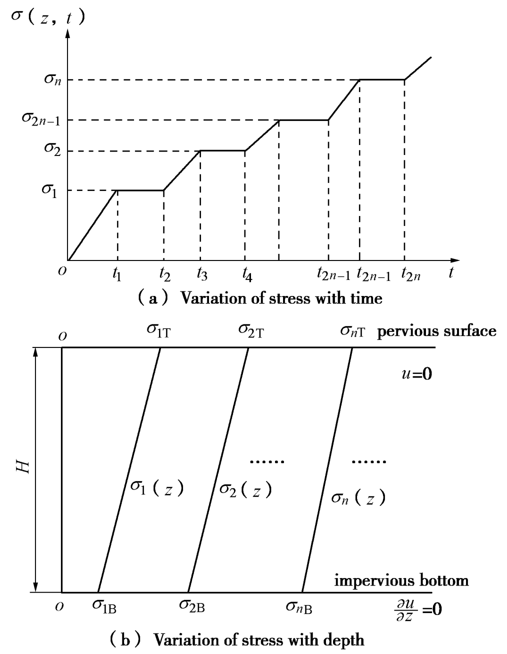

Fig. 1 Multi-stage and depth-dependent stresses within a soil layer due to external load

Following the principle of mass conservation for porous media, the equation governing the one-dimensional consolidation of a saturated soil layer with a multi-stage and depth-dependent stress can be derived as follows:

$$\frac{\partial u}{\partial t}-c_{\mathrm{v}}\frac{\partial^2 u}{\partial z^2}=\frac{\partial\sigma(z,t)}{\partial t}\quad , \tag{3}$$

where u is the excess pore water pressure within the soil layer at any depth and time, and c_{v} is the coefficient of consolidation of the soil layer, which is assumed to be constant during consolidation.

To find the solution to this problem, the top surface is assumed to be fully permeable, whereas the bottom base is assumed to be impervious, as shown in Fig. 1. Therefore, the corresponding boundary conditions can be written as follows:

$$\left.\begin{aligned}&u(z,t)\big|_{z=0}=0\ ,\\&\left.\frac{\partial u(z,t)}{\partial z}\right|_{z=H}=0\ .\end{aligned}\right\} \tag{4}$$

As shown in Fig. 1(a), at the initial moment, the stress due to the external load equals zero, and hence, the excess pore water pressure also equals zero. Therefore, the initial condition is as follows:

$$u(z,t)\big|_{t=0}=0\quad . \tag{5}$$

Referring to the study by Zhu et al.[5] the solution for the excess pore water pressure can be assumed to have the following form by introducing the Fourier series $\sin\left(\frac{M}{H}z\right)$:

$$u(z,t)=\sum_{m=1}^{\infty}T_m(t)\sin\left(\frac{M}{H}z\right)\quad , \tag{6}$$

where $M=\frac{2m-1}{2}\pi,\ (m=1,\ 2,\ 3,\ \cdots)$.

Eq. (6) is satisfied by the boundary conditions in Eq. (4).

Substituting Eq. (6) into Eq. (3) and using the orthogonality of the Fourier series yields:

$$T_m'(t)+\beta_m T_m(t)=Q_m(t)\quad , \tag{7}$$

where $\beta_m=M^2c_{\mathrm{v}}/H^2$, and

$$Q_m(t)=\frac{2}{H}\int_0^H\frac{\partial\sigma(z,t)}{\partial t}\sin\left(\frac{M}{H}z\right)\mathrm{d}z\quad . \tag{8}$$

Combining with Eq. (6), the initial condition in Eq. (5) can be rewritten as

$$T_m(t)\big|_{t=0}=0\quad . \tag{9}$$

Eqs. (7) and (9) are an ordinary differential equation and the corresponding initial condition, respectively. The solution for Eq. (7) satisfying Eq. (9) can be obtained readily as follows:

$$T_m(t)=\mathrm{e}^{-\beta_m t}\int_0^t Q_m(\tau)\mathrm{e}^{\beta_m\tau}\mathrm{d}\tau\quad . \tag{10}$$

The next important procedure is to determine the expression for $Q_m(t)$. By substituting Eq. (1) into Eq. (8), $Q_m(t)$ can be derived as follows:

$$Q_m(t)=\begin{cases}\dfrac{2\left[A_n-(-1)^m\dfrac{B_n}{M}\right]}{M(t_{2n-1}-t_{2n-2})} & (t_{2n-2}\leqslant t\leqslant t_{2n-1}),\\ 0 & (t_{2n-1}\leqslant t\leqslant t_{2n})\end{cases} \tag{11}$$

where $A_n=\sigma_{n\mathrm{T}}-\sigma_{(n-1)\mathrm{T}}$, $B_n=(\sigma_{n\mathrm{B}}-\sigma_{n\mathrm{T}})-(\sigma_{(n-1)B}-\sigma_{(n-1)\mathrm{T}})$.

Substituting Eq. (11) into Eq. (10) yields:

$$T_m(t)=\begin{cases}\displaystyle\sum_{i=1}^{n-1}\left\{\frac{2\left[A_i-(-1)^m\dfrac{B_i}{M}\right]}{M(t_{2i-1}-t_{2i-2})\beta_m}\left(\mathrm{e}^{-\beta_m(t-t_{2i-1})}-\mathrm{e}^{-\beta_m(t-t_{2i-2})}\right)\right\}+\\ \dfrac{2\left[A_n-(-1)^m\dfrac{B_n}{M}\right]}{M(t_{2n-1}-t_{2n-2})\beta_m}\left(1-\mathrm{e}^{\beta_m(t-t_{2n-2})}\right)\quad (t_{2n-2}\leqslant t\leqslant t_{2n-1}).\\ \displaystyle\sum_{i=1}^{n}\frac{2\left[A_i-(-1)^m\dfrac{B_i}{M}\right]}{M(t_{2i-1}-t_{2i-2})\beta_m}\left(\mathrm{e}^{-\beta_m(t-t_{2i-1})}-\mathrm{e}^{-\beta_m(t-t_{2i-2})}\right)\\ \qquad (t_{2n-1}\leqslant t\leqslant t_{2n})\end{cases} \tag{12}$$

By substituting Eq. (12) into Eq. (6), the excess pore water pressure can be finally determined as:

$$u=\begin{cases}\sum_{m=1}^{\infty}\sin\left(\frac{M}{H}z\right)\left\{\sum_{i=1}^{n-1}\frac{2\left[A_i-(-1)^m\frac{B_i}{M}\right](\mathrm{e}^{-\beta_m(t-t_{2i-1})}-\mathrm{e}^{-\beta_m(t-t_{2i-2})})}{M(t_{2i-1}-t_{2i-2})\beta_m}+\frac{2\left[A_n-(-1)^m\frac{B_n}{M}\right]}{M(t_{2n-1}-t_{2n-2})\beta_m}\left(1-e^{\beta_m(t-t_{2n-2})}\right)\right\} & (t_{2n-2}\leqslant t\leqslant t_{2n-1})\\ \sum_{m=1}^{\infty}\sin\left(\frac{M}{H}z\right)\sum_{i=1}^{n}\frac{2\left[A_i-(-1)^m\frac{B_i}{M}\right]}{M(t_{2i-1}-t_{2i-2})\beta_m}(\mathrm{e}^{-\beta_m(t-t_{2i-1})}-\mathrm{e}^{-\beta_m(t-t_{2i-2})}) & (t_{2n-1}\leqslant t\leqslant t_{2n})\end{cases} \tag{13}$$

2 Average degree of consolidation

In this section, the average degree of consolidation is derived for the soil layer based on the solution for the excess pore water pressure. The average degree of consolidation is defined as the ratio of the effective stress at any time to the final total stress over the whole soil layer, i.e.:

$$U(t)=\frac{\int_0^H\left[\sigma(z,t)-u(z,t)\right]\mathrm{d}z}{\int_0^H\sigma_j(z)\mathrm{d}z}\quad , \tag{14}$$

where $U(t)$ is the average degree of consolidation for the soil layer over the whole thickness, and $\sigma(z,t)$ is the total stress due to the external load within the soil at any time as described in Eqs. (1) and (2). If the external load is assumed to be applied in j stages, then the final total stress is equal to $\sigma_j(z)$. Substituting Eqs. (1), (2) and (13) into Eq. (14) yields:

$$U(t)=\begin{cases}\frac{1}{(\sigma_{j\mathrm{B}}+\sigma_{j\mathrm{T}})}\left\{\left[(\sigma_{(n-1)\mathrm{B}}+\sigma_{(n-1)\mathrm{T}})+\frac{(t-t_{2n-2})C_n}{t_{2n-1}-t_{2n-2}}\right]-4\sum_{m=1}^{\infty}\left[\sum_{i=1}^{n-1}\frac{A_i-(-1)^m\frac{B_i}{M}(\mathrm{e}^{-\beta_m(t-t_{2i-1})}-\mathrm{e}^{-\beta_m(t-t_{2i-2})})}{M^2(t_{2i-1}-t_{2i-2})\beta_m}+\frac{A_n-(-1)^m\frac{B_n}{M}}{M^2(t_{2n-1}-t_{2n-2})\beta_m}(1-\mathrm{e}^{\beta_m(t-t_{2n-2})})\right]\right\} & (t_{2n-2}\leqslant t\leqslant t_{2n-1})\\ \frac{1}{(\sigma_{j\mathrm{B}}+\sigma_{j\mathrm{T}})}\left\{(\sigma_{n\mathrm{B}}+\sigma_{n\mathrm{T}})-\sum_{m=1}^{\infty}\sum_{i=1}^{n}\frac{4\left[A_i-(-1)^m\frac{B_i}{M}\right](e^{-\beta_m(t-t_{2i-1})}-e^{-\beta_m(t-t_{2i-2})})}{M^2(t_{2i-1}-t_{2i-2})\beta_m}\right\} & (t_{2n-1}\leqslant t\leqslant t_{2n})\end{cases} \tag{15}$$

where $n\leqslant j,\ C_n=(\sigma_{n\mathrm{B}}+\sigma_{n\mathrm{T}})-(\sigma_{(n-1)\mathrm{B}}+\sigma_{(n-1)\mathrm{T}})$.

In practice, the progress of consolidation settlement of a clayey layer is an issue of interest for engineers and designers. When the average degree of consolidation is obtained, the consolidation settlement of the clay layer can be calculated by the following relation:

$$s_t=s_\infty U(t)\quad , \tag{16}$$

where s_t, s_∞ are the settlements of the whole soil layer at any time and at the final time, respectively.

$$s_\infty=\int_0^H\varepsilon_z\mathrm{d}z=\frac{1}{E_s}\int_0^H\sigma(z,t=\infty)\mathrm{d}z=\frac{1}{E_\mathrm{s}}\int_0^H\sigma_j(z)\mathrm{d}z=\frac{(\sigma_{j\mathrm{B}}+\sigma_{j\mathrm{T}})H}{2E_\mathrm{s}}\quad , \tag{17}$$

here, ε_z is the vertical strain in the soil at any depth, and E_s is the constrained compression modulus of the soil layer.

3 Discussion of solutions

3.1 Case of depth-dependent ramp loading

As a particular case, j = 1 or $\sigma_n(z)=\sigma_1(z)$ ($n=1, 2, 3, \cdots$) indicates the external load is applied gradually from zero to $\sigma_1(z)$ and then holds constant with time. This is called ramp loading or single-stage loading. In addition, when the stress in the soil caused by the above-mentioned ramp loading is assumed to also vary with depth, this type of loading scheme is referred to as depth-dependent ramp loading, following Zhu et al.[5-6] who presented the analytical solutions for a single soil layer subjected to such a load. In the present study, letting j = 1 or $\sigma_n(z)=\sigma_1(z)$, the present solutions for the excess pore water pressure and the average degree of consolidation presented in Eqs. (13) and (15) degenerate into those obtained by Zhu et al.[5] as:

$$u=\begin{cases}\sum_{m=1}^{\infty}\frac{2\left[\sigma_{1\mathrm{T}}-(-1)^m\frac{\sigma_{1\mathrm{B}}-\sigma_{1\mathrm{T}}}{M}\right]}{Mt_1\beta_m}(1-\mathrm{e}^{\beta_m t})\sin\left(\frac{M}{H}z\right) & (0\leqslant t\leqslant t_1)\\ \sum_{m=1}^{\infty}\frac{2\left[\sigma_{1\mathrm{T}}-(-1)^m\frac{\sigma_{1B}-\sigma_{1\mathrm{T}}}{M}\right]}{Mt_1\beta_m}(\mathrm{e}^{-\beta_m(t-t_1)}-\mathrm{e}^{-\beta_m t})\sin\left(\frac{M}{H}z\right) & (t\geqslant t_1)\end{cases} \tag{18}$$

$$U(t)=\begin{cases}\frac{t}{t_1}-\frac{4}{(\sigma_{1\mathrm{B}}+\sigma_{1\mathrm{T}})}\sum_{m=1}^{\infty}\frac{(1-\mathrm{e}^{\beta_m t})}{M^2t_1\beta_m}\cdot\left[\sigma_{1\mathrm{T}}-(-1)^m\frac{\sigma_{1\mathrm{B}}-\sigma_{1\mathrm{T}}}{M}\right] & (0\leqslant t\leqslant t_1)\\ 1-\frac{4}{(\sigma_{n\mathrm{B}}+\sigma_{n\mathrm{T}})}\sum_{m=1}^{\infty}\frac{(\mathrm{e}^{-\beta_m(t-t_1)}-\mathrm{e}^{-\beta_m t})}{M^2t_1\beta_m}\cdot\left[\sigma_{1\mathrm{T}}-(-1)^m\frac{\sigma_{1\mathrm{B}}-\sigma_{1\mathrm{T}}}{M}\right] & (t\geqslant t_1)\end{cases} \tag{19}$$

As evidence of the soundness of our derivation, the solution presented by Zhu et al.[5] reduces to a special case of the solution derived in this study. In addition, letting $\sigma_{1B}=\sigma_{1T}$, Eqs. (18) and (19) can be reduced to the solutions by Olson[4]. Furthermore, letting $\sigma_{1B}=\sigma_{1T}$ and $t_1 \to 0$, Eqs. (18) and (19) can be simplified to Terzaghi's solutions [1]. So, in this sense, the present study presents a more general analytical solution for the consolidation of a single soil layer that encompasses all of the solutions described above.

3.2 Case of two-stage loading

To facilitate the utilization of the present solutions in practical engineering, a particular case of two-stage loading is selected to demonstrate the use of these solutions to calculate the excess pore water pressure and the average degree of consolidation. As stated above, j = 2 indicates that the external load is applied in two stages. Letting j = 2 gives n two values: 1 and 2. When n = 1 (corresponding to time intervals $0 \leqslant t \leqslant t_1$ and $t_1 \leqslant t \leqslant t_2$), the expressions for the excess pore water pressure and the average degree consolidation can be obtained from Eqs. (13) and (15) by letting j = 2 and n = 1. It can then be readily shown that the calculation of excess pore water pressure is the same as that in Eq. (18), whereas the calculation of the average degree of consolidation is as follows:

$$U(t)=\begin{cases}\dfrac{1}{(\sigma_{2B}+\sigma_{2T})}\left[\dfrac{(\sigma_{1B}+\sigma_{1T})t}{t_1}-4\sum_{m=1}^{\infty}\dfrac{\sigma_{1T}-(-1)^m\dfrac{\sigma_{1B}-\sigma_{1T}}{M}}{M^2t_1\beta_m}(1-e^{\beta_m t})\right] & (0\leqslant t\leqslant t_1) \\ \dfrac{1}{(\sigma_{2B}+\sigma_{2T})}\left\{(\sigma_{1B}+\sigma_{1T})-\sum_{m=1}^{\infty}\dfrac{4}{M^2t_1\beta_m}\times(e^{-\beta_m(t-t_1)}-e^{-\beta_m t})\left[\sigma_{1T}-(-1)^m\dfrac{\sigma_{1B}-\sigma_{1T}}{M}\right]\right\} & (t_1\leqslant t\leqslant t_2)\end{cases}, \tag{20}$$

where σ_{1T} and σ_{1B} are the total stresses at the top surface and the bottom of the soil layer caused by the first-stage loading.

When n = 2 (corresponding to the time intervals of $t_2 \leqslant t \leqslant t_3$ and $t \geqslant t_3$), by letting j = 2 and n = 2, the detailed expressions for the excess pore water pressure and average degree of consolidation can be derived from Eqs. (13) and (15) as

$$u=\begin{cases}\sum_{m=1}^{\infty}\sin\left(\dfrac{M}{H}z\right)\left\{\dfrac{2\left[A_1-(-1)^m\dfrac{B_1}{M}\right]}{Mt_1\beta_m}(e^{-\beta_m(t-t_1)}-e^{-\beta_m t})+\dfrac{2\left[A_2-(-1)^m\dfrac{B_2}{M}\right]}{M(t_3-t_2)\beta_m}(1-e^{\beta_m(t-t_2)})\right\} & (t_2\leqslant t\leqslant t_3) \\ \sum_{m=1}^{\infty}\sin\left(\dfrac{M}{H}z\right)\left\{\dfrac{2\left[A_1-(-1)^m\dfrac{B_1}{M}\right]}{Mt_1\beta_m}(e^{-\beta_m(t-t_1)}-e^{-\beta_m t})+\dfrac{2\left[A_2-(-1)^m\dfrac{B_2}{M}\right]}{M(t_3-t_2)\beta_m}(e^{-\beta_m(t-t_3)}-e^{-\beta_m(t-t_2)})\right\} & (t\geqslant t_3)\end{cases}. \tag{21}$$

$$U(t)=\begin{cases}\dfrac{1}{(\sigma_{2B}+\sigma_{2T})}\left\{\left[(\sigma_{1B}+\sigma_{1T})+\dfrac{t-t_2}{t_3-t_2}C_2\right]-4\sum_{m=1}^{\infty}\left[\dfrac{A_1-(-1)^m\dfrac{B_1}{M}}{M^2t_1\beta_m}(e^{-\beta_m(t-t_1)}-e^{-\beta_m t})+\dfrac{A_2-(-1)^m\dfrac{B_2}{M}}{M^2(t_3-t_2)\beta_m}(1-e^{\beta_m(t-t_2)})\right]\right\} & (t_2\leqslant t\leqslant t_3) \\ 1-\dfrac{4}{(\sigma_{2B}+\sigma_{2T})}\sum_{m=1}^{\infty}\left\{\dfrac{\left[A_1-(-1)^m\dfrac{B_1}{M}\right]}{M^2t_1\beta_m}\cdot(e^{-\beta_m(t-t_1)}-e^{-\beta_m t})+\dfrac{\left[A_2-(-1)^m\dfrac{B_2}{M}\right]}{M^2(t_3-t_2)\beta_m}\cdot(e^{-\beta_m(t-t_3)}-e^{-\beta_m(t-t_2)})\right\} & (t\geqslant t_3)\end{cases}. \tag{22}$$

where σ_{2T} and σ_{2B} are the total stresses at the top surface and the bottom of the soil layer caused by the second-stage loading; $A_1=\sigma_{1T}$, $B_1=\sigma_{1B}-\sigma_{1T}$, $A_2=\sigma_{2T}-\sigma_{1T}$, $B_2=\sigma_{2B}-\sigma_{2T}-(\sigma_{1B}-\sigma_{1T})$, $C_2=\sigma_{2B}+\sigma_{2T}-(\sigma_{1B}+\sigma_{1T})$.

4 Consolidation behavior with multi-stage and depth-dependent stress

In this section, the consolidation behavior of a single soil layer with multi-stage and depth-dependent stress is investigated. For more general significance, the vertical time factor T_v （$T_v = c_v t / H^2$） is selected as the time axis for all of the figures instead of the real time t. The time factors corresponding to the loading periods t_1, t_2, and t_3 are T_1 ($T_1 = c_v t_1 / H^2$), T_2 ($T_2 = c_v t_2 / H^2$) and T_3 ($T_3 = c_v t_3 / H^2$), respectively.

Fig. 2 shows the variation of the excess pore water pressure with time at various depths. It can be seen that the excess pore water pressure increases during the loading stage (i.e., in the time interval [t_{2n-2}, t_{2n-1}]), whereas it decreases during the constant loading stage (i.e., in the time interval [t_{2n-1}, t_{2n}]). In addition, the excess pore water pressure increases with the increasing depth because the drainage boundary is set at the top surface; therefore, larger stress values at locations close to the top surface lead to a more rapid dissipation of the excess pore water pressure.

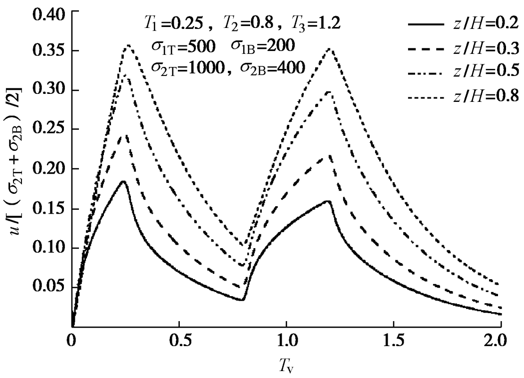

Fig. 2 Variation of excess pore water pressure with time at different depths

Fig. 3 shows a comparison among the three cases with various loading schemes, including the case of instantaneous loading proposed by Terzaghi[14], the case of single-stage loading (i.e., ramp loading) given by Zhu et al.[5] and the case of two-stage loading used in this study. To obtain the equivalent conditions, the stress is assumed to be uniform with depth for all of the cases, that is, $\sigma_{nB} = \sigma_{nT}$. Obviously, the consolidation rate predicted by Terzaghi[14] is the most rapid, the one predicted by the present solution is the slowest, and the one predicted by Zhu et al.[5] is in the middle. This order is because the loading rate of Terzaghi's solution [14] is faster than that in the other two solutions, and the loading rate of Zhu et al.[5] solution is in turn faster than that in the present solution. In other words, the consolidation rate is accelerated by an increase in the loading rate.

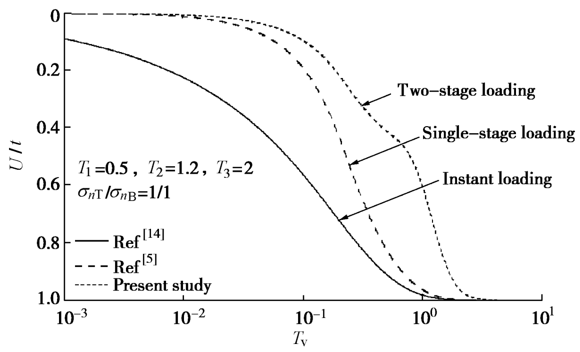

Fig. 3 Comparison among several loading schemes

Fig. 4 shows the influence of the stress distribution along the thickness on the consolidation rate of the single soil layer. It can be seen that a larger value of the ratio of the top to bottom stress results in a more rapid consolidation. The reason for this, as stated earlier, is that the drainage boundary is set at the top surface of the soil layer. Therefore, in practice, if a rapid consolidation rate is desired, a large stress at the location close to the drainage boundary is preferred.

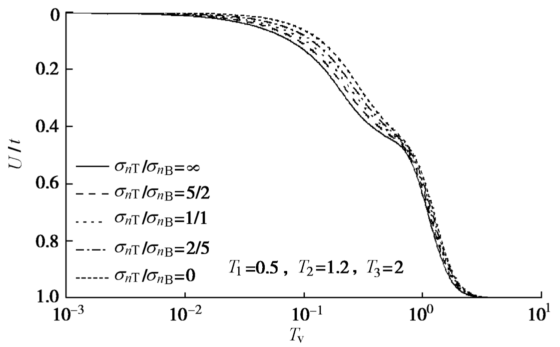

Fig. 4 Average degrees of consolidation with different stress distributions with depth

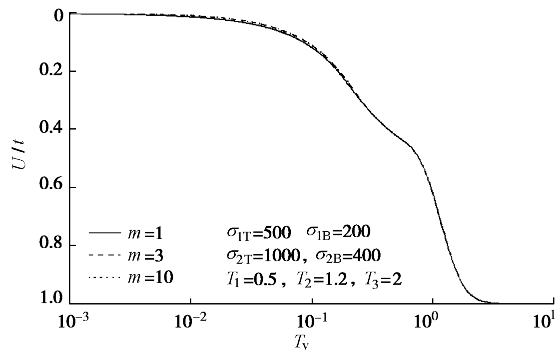

Fig. 5 Convergence of solution with different calculation terms

To facilitate practicing engineers in determining the proper calculation in terms of achieving sufficient accuracy, the results of applying various calculation terms are presented in Fig. 5. It can be seen here that the average degree of consolidation calculated with 3 terms and that with 10 terms coincide well, indicating that in practical engineering, at most three terms of the series of the present solution will yield sufficient accuracy in calculating the average degree of consolidation.

5 Conclusions

A thorough, explicit analytical solution is obtained for the consolidation of a single soil layer with multi-stage and depth-dependent stress. The present solution can be reduced to several previously reported solutions. For engineers, the present solution is very simple to use because the series solution converges very rapidly and at most three calculation terms yield sufficient accuracy in predicting the average degree of consolidation. The results of the parameter analysis on the consolidation behavior show that the excess pore water pressure increases during the loading stage, whereas it decreases during a constant loading stage. In addition, the excess pore water pressure increases with the increasing depth. For a soil layer with a fully impermeable top surface and an impervious base, a larger value of the ratio of the top to bottom stress results in a more rapid consolidation, i.e., the consolidation rate accelerates with the increase of the loading rate.

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